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A QUASI-REVERSIBILITY APPROACH TO SOLVE THE INVERSE OBSTACLE PROBLEM

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Abstract. We introduce a new approach based on the coupling of the method of quasi-reversibility and a simple level set method in order to solve the inverse obstacle problem with Dirichlet boundary condition. We provide a theoretical justification of our approach and illustrate its feasibility with the help of numerical experiments in $2D$.

1. **Introduction.** In this paper, we address the inverse obstacle problem, defined as follows. Let $\mathcal{D}$ be an open, bounded and connected domain of $\mathbb{R}^N$, with Lipschitz boundary. Let $\mathcal{O} \Subset D$ be an open domain with a continuous boundary, referred to as the obstacle, and such that $\Omega := \mathcal{D} \setminus \overline{\mathcal{O}}$ is connected. Let Γ be an open subset of $\partial\mathcal{D}$. Given a pair of data $(g_0, g_1) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ with $(g_0, g_1) \neq (0, 0)$, the inverse obstacle problem consists in finding a domain $\mathcal{O}$ and a function $u \in H^1(\Omega) \cap C^0(\overline{\Omega})$ which satisfies

$$(1) \qquad \begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = g_0 & \text{on } \Gamma \\ \partial_n u = g_1 & \text{on } \Gamma \\ u = 0 & \text{on } \partial\mathcal{O}. \end{cases}$$

We first recall the following uniqueness result concerning our inverse obstacle problem.

Theorem 1.1. *The domain $\mathcal{O}$ and the function u that satisfy (1) are uniquely defined by data (g_0, g_1).*

The proof of that classical theorem is a slight adaptation of the proof given in [23] (theorem 5.1) for the same problem with Helmholtz operator instead of Laplace operator. This proof is based on the lemma 2.2 given hereafter, in particular the fact that $u \in C^0(\overline{\Omega})$ implies that no regularity is required for obstacle $\mathcal{O}$.

Theorem 1.1 means that we can reasonably try to retrieve the unknown obstacle $\mathcal{O}$ and the unknown function u from the Cauchy data (g_0, g_1) on Γ, which is the

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objective of the article. However, it should be noted that since in practice the Cauchy data are known from measurements, they are corrupted by some noise of amplitude δ. Thus we have to cope with some contaminated data (g_0^δ, g_1^δ) rather than (g_0, g_1).

There is a huge literature on that kind of problem, and many approaches have been proposed. Some of them are based on a parameterization of the obstacle, in the spirit of [23] (chapter 5.3), some others on shape sensitivity, in the spirit of [32, 18] or topological gradient like in [13, 20]. In the particular case of the Laplace equation in $2D$, some methods based on conformal mappings were also performed, like in [14]. Among all articles based on such methods, the specific case of the obstacle characterized by a homogeneous Dirichlet data is addressed in [14] and in [15, 16]. Another successful approach consists of level set techniques, which transform the problem of finding a geometry into the problem of finding the level set 0 of a function. Since their introduction in [8], the level set techniques have been extensively used in the framework of inverse problems, mainly because they can handle topological changes. This is illustrated for the inverse obstacle problem for example in [21, 12, 19, 11]. In all these works, a minimization problem is solved, and the level set function is the solution of a Hamilton-Jacobi equation, the advection velocity of which is associated to the shape derivative of the cost function.

Our approach consists in coupling the method of quasi-reversibility and a level set technique in order to identify the pair $(\mathcal{O}, u)$. The method of quasi-reversibility is used to provide an approximate solution to the so-called Cauchy problem, which consists, for fixed obstacle $\mathcal{O}$, to find the solution in Ω to the three first equations of problem (1), given the data (g_0, g_1) on Γ. Such problem is known to be ill-posed, that is small errors on the data (g_0, g_1) produce large errors on the solution u. The method of quasi-reversibility for elliptic equation, first introduced in [30] and revisited in [6, 1, 2], provides a regularized solution of the Cauchy problem. It consists in transforming the ill-posed second-order problem into a family of well-posed fourth-order problems. This family depends on a small parameter ε in such a way that the regularized solution u_ε tends to the "exact" solution u when ε tends to 0. Two particular questions have been studied : the first one concerns the convergence rate when $\varepsilon \to 0$, the second one concerns, when the data (g_0, g_1) are corrupted by some noise of amplitude δ, the effective choice of ε as a function of δ. Some answers to the first question are given in [6, 3, 4], while some answers to the second one are given in [2, 5, 9]. A few results concerning these two questions will be recalled in this paper. Let us remark that the method of quasi-reversibility allows us to approximate the solution u once its domain Ω is known. However, for our inverse obstacle problem (1), the domain $\Omega = \mathcal{D} \setminus \overline{\mathcal{O}}$ is also unknown, since the obstacle $\mathcal{O}$ is unknown. This is precisely the problem that is raised in [26] (chapter 5), in the context of the identification of a plasma boundary from outer magnetic measurements. However, in [26], it is assumed that the solution u can be extended outside Ω in the sense of $\Delta u = 0$, which of course significantly simplifies the problem but is not correct in general. In order to get rid of such assumption, we introduce a level set technique in order to identify $\mathcal{O}$ as the set $\{x \in \mathcal{D},\, \phi(x) \leq 0\}$, where ϕ is a function that is computed with the help of the quasi-reversibility solution u_ε we have introduced before. As it is proposed in [21, 12, 19, 11], one could have obtained ϕ as the solution of an eikonal equation. This is of course feasible. Here, we introduce a much simpler level set technique based on the computation of a non-homogeneous Dirichlet problem for the Laplace equation with appropriate second member, the solution of which is ϕ. We hence obtain an iterative approach in which u_ε and ϕ are updated alternatively, so that when the number of iterations goes to infinity, the set $\{x \in \mathcal{D}, \phi(x) < 0\}$ provides an approximation of $\mathcal{O}$ while u_ε provides an approximation of u in Ω. It should be noted that our approach is original in the sense that it is not based on an optimization procedure. In this sense, our study could be compared to the treatment of a Bernoulli problem in [10], in which a well-posed problem is solved to update the solution u of the problem for fixed domain, while the level set function ϕ that defines the domain is updated by solving a simple time-dependent equation. However, in the case of the inverse obstacle problem, the Cauchy data apply to the known boundary while in the case of the Bernoulli problem, the Cauchy data apply to the unknown one.

Our paper is organized as follows. In section 2, we introduce our level set technique and provide its justification. In section 3, we briefly describe the method of quasi-reversibility as well as its justification. In particular, we present a discretization of the method based on nonconforming finite elements. A proof of convergence for our finite element method is postponed in an appendix (section 7). Section 4 describes our approach obtained by coupling quasi-reversibility and level set methods. Numerical experiments are presented in section 5, showing the feasibility of our method. We complete our study with a few concluding remarks in section 6.

2. **About a simple level set method.**

We present our level set method and show it enables us to identify the obstacle $\mathcal{O}$ provided the function u that solves (1) is known. It is not true in practice, because the Cauchy problem is ill-posed and must be computed from some noisy data (g_0^δ, g_1^δ). Such problem is addressed in the next section.

We consider the notations of the introduction and we consider a function $\tilde{u}$ in the whole domain $\mathcal{D}$ which satisfies:

$$\left\{\begin{array}{rcll} \tilde{u} & = & |u| & \text{in } \Omega \\ \tilde{u}|_{\mathcal{O}} & \in & H_0^1(\mathcal{O}) & \\ \tilde{u} & \leq & 0 & \text{in } \mathcal{O}. \end{array}\right. \tag{2}$$

Such functions $\tilde{u}$ exist (take simply $\tilde{u} = 0$ in $\mathcal{O}$) and belong to $H^1(\mathcal{D})$. Let us verify this fact. First, the fact that $u \in H^1(\Omega)$ implies that

$$|u| \in H^1(\Omega), \tag{3}$$

by using the following more comprehensive lemma, which is proved in [29] (corollary 3.1.12).

Lemma 2.1. *Let Ω be an open domain of $\mathbb{R}^N$, and u, v two functions in $H^1(\Omega)$ (resp. $H_0^1(\Omega)$). Then $|u|$, $\inf(u,v)$, $\sup(u,v) \in H^1(\Omega)$ (resp $\in H_0^1(\Omega)$). Furthermore, the mappings $u \mapsto |u|$ $(H^1(\Omega) \to H^1(\Omega))$, $(u,v) \mapsto \inf(u,v)$ and $(u,v) \mapsto \sup(u,v)$ $(H^1(\Omega) \times H^1(\Omega) \to H^1(\Omega))$ are continuous.*

Then the fact that $|u| \in H^1(\Omega) \cap C^0(\overline{\Omega})$ and $u = 0$ on $\partial\mathcal{O}$ implies that

$$\phi|u| \in H_0^1(\Omega), \quad \forall \phi \in C_0^\infty(\mathcal{D}). \tag{4}$$

This results from the following lemma, which is proved in [25] (see theorem IX.17 and remark 20).

Lemma 2.2. *Let Ω denote an open subset of $\mathbb{R}^N$, and $u \in H^1(\Omega) \cap C^0(\overline{\Omega})$ such that $u = 0$ on $\partial\Omega$. Then $u \in H_0^1(\Omega)$, where $H_0^1(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$.*

Lastly, (3), (4) and $\tilde{u}|_{\mathcal{O}} \in H_0^1(\mathcal{O})$ imply that $\tilde{u} \in H^1(\mathcal{D})$, whence $\Delta\tilde{u} \in H^{-1}(\mathcal{D})$.

We now define a sequence of open domains ω_m by following induction. Let us choose $f \in H^{-1}(\mathcal{D})$ such that

$$f \geq \Delta\tilde{u} \tag{5}$$

in the sense of $H^{-1}(\mathcal{D})$, and an open domain ω_0 such that $\mathcal{O} \subset \omega_0 \Subset \mathcal{D}$. The open domain ω_m being given, we define

$$\omega_{m+1} = \omega_m \setminus \operatorname{supp}(\sup(v_{f,\omega_m}, 0)), \tag{6}$$

where $v_{f,\omega}$ is the unique solution $v \in H^1(\omega)$ of the non-homogeneous Dirichlet problem

$$\left\{ \begin{array}{ccc} \Delta v & = & f \\ v - \tilde{u} & \text{in} & H_0^1(\omega). \end{array} \right. \tag{7}$$

Let us remark that problem (7) is equivalent to find $w_{f,\omega}$ as the unique solution $w \in H_0^1(\omega)$ of the homogeneous Dirichlet problem

$$\Delta w = f - \Delta\tilde{u} \tag{8}$$

with $f - \Delta\tilde{u} \in H^{-1}(\omega)$.

Our objective is to prove that under some additional assumption, the sequence of domains ω_m converge in a certain sense to the obstacle $\mathcal{O}$. This result mainly relies on the weak maximum principle, which is proved for example in [24] (paragraph 8.1).

Proposition 1. *Let Ω be an open domain of $\mathbb{R}^N$, and $u \in H^1(\Omega)$ such that $\Delta u \geq 0$ in the sense of $H^{-1}(\Omega)$, and $\sup(u, 0) \in H_0^1(\Omega)$. Then $\sup(u, 0) = 0$ in Ω.*

We begin with the following proposition.

Proposition 2. *The sequence of domains ω_m satisfies: for all $m \in \mathbb{N}$, $\mathcal{O} \subset \omega_{m+1} \subset \omega_m \Subset \mathcal{D}$.*

Proof. Let us denote $\omega_m = \omega$, such that $\mathcal{O} \subset \omega \Subset \mathcal{D}$. We have to prove that $\tilde{\omega} := \omega_{m+1}$ satisfies $\mathcal{O} \subset \tilde{\omega}$. By using the weak maximum principle and the fact that $f - \Delta\tilde{u} \geq 0$, we obtain $w_{f,\omega} \leq 0$ in ω, that is $v_{f,\omega} = w_{f,\omega} + \tilde{u} \leq \tilde{u}$ in ω. Since $\tilde{u} \leq 0$ in $\mathcal{O} \subset \omega$, we obtain $v_{f,\omega} \leq 0$ in $\mathcal{O}$. Hence $\mathcal{O} \subset \omega \setminus \operatorname{supp}(\sup(v_{f,\omega}, 0)) = \tilde{\omega}$. The proof is complete. □

Since $\forall m \in \mathbb{N}$, $\omega_{m+1} \subset \omega_m \subset \mathcal{D}$, with $\mathcal{D}$ an open bounded domain of $\mathbb{R}^N$, we immediately obtain the following proposition by using [29] (paragraph 2.2.3).

Proposition 3. *The sequence of open domains ω_m converges, in the sense of the Hausdorff distance for open domains, to the set*

$$\omega = \overbrace{\bigcap_m \omega_m}^{\circ},$$

such that $\mathcal{O} \subset \omega \Subset \mathcal{D}$.

The definition of the Hausdorff distance for open domains and various relative properties are detailed in [29] (paragraph 2.2.3).

Lastly, we state the main theorem of this section. In this view, we consider the following assumption, which concerns the continuity of Dirichlet solution of the Laplace equation with respect to the domain and is extensively analyzed in [29].

Such assumption is discussed at the end of this section.

Assumption [H] : if $w_{f,\omega}$ denotes the solution of the homogeneous problem (8) in the domain ω, the sequence w_{f,ω_m} tends to $w_{f,\omega}$ in $H^1_0(\mathcal{D})$ when $m \to +\infty$.

In order to obtain our main theorem, we also need the two following lemmas. The first one is proved in [25] (lemma IX.5).

Lemma 2.3. *Let Ω be an open domain of $\mathbb{R}^N$, and $u \in H^1(\Omega)$ such that* $\mathrm{supp}(u)$ *is a compact set included in Ω. Then $u \in H^1_0(\Omega)$.*

Lemma 2.4. *Let Ω be an open domain of $\mathbb{R}^N$, and $u \in H^1(\Omega)^+$ ($u \in H^1(\Omega)$ and $u \geq 0$ a.e. in Ω). If there exists $v \in H^1_0(\Omega)$ such that $u \leq v$ a.e. in Ω, then $u \in H^1_0(\Omega)$.*

Proof. Since $v \in H^1_0(\Omega)$, there exists a sequence of functions $v_m \in C^\infty_0(\Omega)$ such that $v_m \to v$ in $H^1(\Omega)$. With the help of lemma 2.1, we have $w_m := \inf(v_m, u) \to \inf(v, u) = u$ in $H^1(\Omega)$. Since $u \geq 0$ in Ω and $\mathrm{supp}(v_m)$ is compact in Ω, $\mathrm{supp}(w_m)$ is compact in Ω and lemma 2.3 implies $w_m \in H^1_0(\Omega)$. Since $H^1_0(\Omega)$ is as close subspace of $H^1(\Omega)$, it follows that $u \in H^1_0(\Omega)$. □

We now state our main theorem, which establishes under assumption [H] the convergence of the sequence of ω_m to the obstacle $\mathcal{O}$ in the sense of Hausdorff distance for open domains.

Theorem 2.5. *We consider the domains $\mathcal{D}$, $\mathcal{O}$, Γ, and the function u as defined in the introduction. Let $\tilde{u} \in H^1(\mathcal{D})$ and $f \in H^{-1}(\mathcal{D})$ satisfy (2) and (5).*

Let ω_0 denote an open domain such that $\mathcal{O} \subset \omega_0 \Subset D$, as well as the decreasing sequence of open domains ω_m defined by (6) and (7).

With additional assumption [H], we have

$$\overbrace{\bigcap_m \omega_m}^{\circ} = \mathcal{O},$$

with convergence in the sense of Hausdorff distance for open domains.

Proof. We already know that $\mathcal{O} \subset \omega$. If we assume that $\mathcal{O} \neq \omega$, let us denote $\mathcal{R} = \omega \setminus \overline{\mathcal{O}}$. We shall find a contradiction. The assumption [H] implies that $w_{f,\omega_m} \to w_{f,\omega}$ in $H^1_0(\mathcal{D})$, and hence also in $H^1(\omega)$. Then

$$\|v_{f,\omega} - v_{f,\omega_m}\|_{H^1(\omega)} = \|w_{f,\omega} + \tilde{u} - w_{f,\omega_m} - \tilde{u}\|_{H^1(\omega)} \overset{m\to\infty}{\longrightarrow} 0,$$

that is $v_{f,\omega_m} \to v_{f,\omega}$ in $H^1(\omega)$. Since $\omega \subset \omega_m$ for all m, we have $v_{f,\omega_m} \leq 0$ a.e. in ω, end hence $v_{f,\omega} \leq 0$ a.e. in ω.

Now let us prove that $\tilde{u} \in H^1_0(\mathcal{R})$. We recall that $\tilde{u} - v_{f,\omega} \geq 0$ in ω, that is $\tilde{u} - v_{f,\omega} \in H^1_0(\omega)^+$. By using corollary 3.1.13 in [29], there exists a sequence $\psi_m \in C^\infty_0(\omega)^+$ such that $\psi_m \to \tilde{u} - v_{f,\omega}$ in $H^1(\omega)$. Let $\phi \in C^\infty_0(\mathcal{D})^+$, $\phi \equiv 1$ on $\overline{\omega}$. We have $\phi\tilde{u} \in H^1_0(\mathcal{D} \setminus \overline{\mathcal{O}})^+$. As a consequence there exists $\phi_m \in C^\infty_0(\mathcal{D} \setminus \overline{\mathcal{O}})^+$ such that $\phi_m \to \phi\tilde{u}$ in $H^1(\mathcal{D} \setminus \overline{\mathcal{O}})$. Now, by using lemma 2.1, the functions $\theta_m := \inf(\phi_m|_{\mathcal{R}}, \psi_m|_{\mathcal{R}})$ converge to $\inf(\phi\tilde{u}|_{\mathcal{R}}, (\tilde{u} - v_{f,\omega})|_{\mathcal{R}}) = \tilde{u}|_{\mathcal{R}}$ in $H^1(\mathcal{R})$, by using $v_{f,\omega} \leq 0$ a.e. in ω and $\phi \equiv 1$ on ω.

Let us denote K_m and L_m the supports of ϕ_m and ψ_m respectively. We have $K_m \subset \mathcal{D} \setminus \overline{\mathcal{O}}$ and $L_m \subset \omega$, hence $K_m \cap L_m \subset \mathcal{R}$. For $x \in \mathcal{R} \setminus (K_m \cap L_m)$, either $\phi_m(x) = 0$ and $\psi_m(x) \geq 0$, or $\phi_m \geq 0$ and $\psi_m = 0$, hence $\theta_m(x) = 0$. This implies that $\text{supp}(\theta_m) \subset K_m \cap L_m$, that is θ_m in compactly supported in $\mathcal{R}$. Since θ_m converges to $\tilde{u}$ in $H^1(\mathcal{R})$, $\tilde{u} \in H^1_0(\mathcal{R})$.

It remains to prove that $u \in H^1_0(\mathcal{R})$. Let us remark that $\sup(u,0), \sup(-u,0) \in H^1(\mathcal{R})$ satisfy $0 \leq \sup(u,0), \sup(-u,0) \leq \tilde{u} = |u|$ in $\mathcal{R}$, then by using lemma 2.4 we obtain $\sup(u,0), \sup(-u,0) \in H^1_0(\mathcal{R})$, and $u = \sup(u,0) - \sup(-u,0) \in H^1_0(\mathcal{R})$.

Since $\Delta u = 0$ in $\mathcal{R}$ and $\mathcal{D} \setminus \overline{\mathcal{O}} = \Omega$ is connected, $u = 0$ in Ω from unique continuation, which contradicts the fact that $(g_0, g_1) \neq (0,0)$. We conclude that $\omega \setminus \overline{\mathcal{O}} = \emptyset$. As a conclusion, $\mathcal{O} \subset \omega \subset \overline{\mathcal{O}}$. Since $\mathcal{O}$ has a continuous boundary, the interior of the set $\overline{\mathcal{O}}$ is $\mathcal{O}$, and lastly $\mathcal{O} = \omega$. ☐

Now let us discuss the assumption [H].

In this view, we introduce a new definition and three other assumptions.

Definition : we say that the open set ω has the cone property with (θ, r) if by introducing the finite open cone

$$C(y,\xi,\theta,r) = \{z \in \mathbb{R}^N,\, (z-y,\xi) > \cos(\theta)|z-y|, \quad 0 < |z-y| < r\},$$

for all $x \in \partial\omega$, there exists ξ_x with $|\xi_x| = 1$ such that for all $y \in \overline{\omega} \cap B(x,r)$, then $C(y,\xi_x,\theta,r) \subset \omega$.

As proved in [29] (theorem 2.4.7), the above definition for bounded open domain ω is equivalent to the fact that ω has a Lipschitz boundary.

Assumption [H1] : $f \geq 0$ in the sense of $H^{-1}(D)$.

Assumption [H2] : the functions v_{f,ω_m} belong to $C^0(\overline{\omega_m})$ for all $m \in \mathbb{N}$.

Assumption [H3] : the domains ω_m satisfy the cone property with $(\varepsilon, \varepsilon)$ for all $m \in \mathbb{N}$, and $\varepsilon > 0$ independent of m.

We analyze two different cases for which assumption [H] holds: a case in the two dimensional setting ($N = 2$) and a case with no restriction on dimension ($N \geq 2$). This takes the form of proposition 4 and proposition 5. These two propositions are based on Šverak's theorem for $N = 2$ (see [29], theorem 3.4.14) and theorem 3.2.13 in [29] for $N \geq 2$, which are stated below.

Theorem 2.6. *Let $\mathcal{D}$ be an open domain of $\mathbb{R}^2$ and $g \in H^{-1}(\mathcal{D})$. For open domain $\omega \subset \mathcal{D}$, $u_{g,\omega}$ denotes the unique function $u \in H^1_0(\omega)$ which solves $\Delta u = g$ in ω.*

Let ω_m be a sequence of open domains such that $\mathcal{D} \setminus \overline{\omega_m}$ is connected and which converges to ω in the sense of Hausdorff distance for open domains. Then u_{g,ω_m} converges to $u_{g,\omega}$ in $H^1_0(\mathcal{D})$.

Theorem 2.7. *Let $\mathcal{D}$ be an open domain of $\mathbb{R}^N$ ($N \geq 2$) and $g \in H^{-1}(\mathcal{D})$. For open domain $\omega \subset \mathcal{D}$, $u_{g,\omega}$ denotes the unique function $u \in H^1_0(\omega)$ which solves $\Delta u = g$ in ω.*

Let ω_m be a sequence of open domains which satisfy assumption [H3] and which converges to ω in the sense of Hausdorff distance for open domains. Then u_{g,ω_m} converges to $u_{g,\omega}$ in $H^1_0(\mathcal{D})$.

Proposition 4. *For $N = 2$, if $\mathcal{D} \setminus \overline{\omega_0}$ is connected, the assumptions [H1] and [H2] imply assumption [H].*

Proof. In view of proposition 3 and theorem 2.6 with $g = f - \Delta\tilde{u}$, we simply have to prove that if assumptions [H1] and [H2] are satisfied and $\mathcal{D} \setminus \overline{\omega_0}$ is connected, then the open domains $\mathcal{D} \setminus \overline{\omega_m}$ are connected for all $m \in \mathbb{N}$. Assume that $\omega := \omega_m$ is such that $\mathcal{D} \setminus \overline{\omega_m}$ is connected. Let us denote $\tilde{\omega} := \omega_{m+1}$, UB the unbounded connected component of $\mathbb{R}^N \setminus \overline{\tilde{\omega}}$ and $B = \mathbb{R}^N \setminus \overline{UB}$.

We first prove that $\tilde{\omega} \subset B \subset \omega$. To obtain the first inclusion, we remark that $UB \subset \mathbb{R}^N \setminus \overline{\tilde{\omega}}$, hence $\tilde{\omega} \subset \mathbb{R}^N \setminus \overline{UB} = B$. To obtain the second inclusion, we remark that $\tilde{\omega} \subset \omega$, that is $\mathbb{R}^N \setminus \overline{\omega} \subset \mathbb{R}^N \setminus \overline{\tilde{\omega}}$. The open set $\mathbb{R}^N \setminus \overline{\omega}$ is connected and unbounded, it follows that $\mathbb{R}^N \setminus \overline{\omega} \subset UB$, and lastly $B = \mathbb{R}^N \setminus \overline{UB} \subset \omega$.

Assumption [H1] implies $\Delta v_{f,\omega} = f \geq 0$ in ω, whence $\Delta v_{f,\omega} \geq 0$ in B. On the other hand, $\sup(v_{f,\omega}, 0) = 0$ in $\tilde{\omega}$. Assumption [H2] implies then that $v_{f,\omega} \in C^0(\overline{\omega})$, so $\sup(v_{f,\omega}, 0) \in C^0(\overline{\omega})$. It follows that $\sup(v_{f,\omega}, 0) = 0$ on $\partial\tilde{\omega}$, and since $\partial B \subset \partial\tilde{\omega}$, we have $\sup(v_{f,\omega}, 0) = 0$ on ∂B. With the help of lemma 2.1 and lemma 2.2, we conclude that $\sup(v_{f,\omega}, 0) \in H^1_0(B)$. Following the weak maximum principle in B (see proposition 1), we obtain $\sup(v_{f,\omega}, 0) = 0$ in B, and hence $B \subset \tilde{\omega}$, that is $\tilde{\omega} = B$, and hence $D \setminus \overline{\tilde{\omega}} = D \cap UB$, which is connected. □

Remark 1. From the proof of proposition 4, we deduce that whatever $N \geq 2$, if $\mathcal{D} \setminus \overline{\omega_0}$ is connected and assumptions [H1] [H2] hold, then the open domains $\mathcal{D} \setminus \overline{\omega_m}$ are connected for all m.

Proposition 5. *The assumption [H3] implies assumption [H].*

The proof is an immediate consequence of proposition 3 and theorem 2.7 with $g = f - \Delta\tilde{u}$.

In the following remarks, we now discuss the assumptions [H1] and [H2].

Remark 2. Concerning the assumption [H1], let us recall (see [31], chapter V, paragraph 4) that the positive distributions $f \in H^{-1}(\mathcal{D})$ are Radon measures, that is they belong to $C^0_0(\mathcal{D})'$. Then assumption [H1] combined with (5) imply that $\Delta\tilde{u} \in C^0_0(\mathcal{D})'$. Conversely, if we assume that $\Delta\tilde{u} \subset C^0_0(\mathcal{D})'$, it is not clear that we can find $f \in H^{-1}(\mathcal{D})$ which satisfies both $f \geq \Delta\tilde{u}$ and $f \geq 0$ in the sense of $H^{-1}(\mathcal{D})$.

But if furthermore $\Delta\tilde{u} \in L^2(\mathcal{D})$, then we can simply choose $f \geq \sup(\Delta\tilde{u}, 0)$, so there exists $f \in L^2(\mathcal{D})$ which satisfies both assumption [H1] and (5). If $\Delta\tilde{u} \in L^\infty(\mathcal{D})$, then $\sup(\Delta\tilde{u}, 0) \in L^\infty(\mathcal{D})$, hence f may be chosen as a positive constant in $\mathcal{D}$.

Remark 3. Now we consider the assumption [H2]. For $N = 2, 3$, if we assume that $f \in L^2(\mathcal{D})$ (see remark 2), and the domains ω_m have a Lipschitz boundary for all $m \in \mathbb{N}$, then in virtue of standard regularity for problem (7) (see theorem 8.30 in [24]), we have $v_{f,\omega_m} \in C^0(\overline{\omega_m})$, that is assumption [H2] is satisfied.

To conclude this section, theorem 2.5 suggests a level set method in order to retrieve the obstacle $\mathcal{O}$. It consists in solving, starting from the initial guess ω_0, the non-homogeneous Dirichlet problem (7) in ω_0 with sufficiently large second member f, then in selecting the subdomain ω_1 of ω_0 such that the obtained solution satisfies $v \leq 0$ in ω_1, and so on. The domains ω_m converge to the obstacle $\mathcal{O}$. However, we keep in mind that the non-homogeneous Dirichlet condition in (7) is $|u|$, which is unknown since u is unknown. More precisely, finding u from the Cauchy data (g_0, g_1) is an ill-posed problem, and these Cauchy data are noisy. This is the reason why we focuss now our interest on a regularization process to calculate, in a stable manner, a quasi-solution u_ε which is close to u in a certain sense. The method of quasi-reversibility is such a regularization process. It should be noted that if we replace $\tilde{u}$ by some $\tilde{u}_\varepsilon$ in (7), with $\tilde{u}_\varepsilon$ different from $|u|$ outside $\mathcal{O}$, it is clear that since for all m, $\omega_{m+1} \subset \omega_m \Subset \mathcal{D}$, the sequence of open domains ω_m is still convergent (in the sense of the Hausdorff distance for open domains) to a ω. Moreover, if we assume that $f - \Delta\tilde{u}_\varepsilon \geq 0$ in $\mathcal{D}$ and $\tilde{u}_\varepsilon \leq 0$ in $\mathcal{O}$, then $\mathcal{O} \subset \omega$ (see the proof of proposition 2). However, theorem 2.5 is not applicable any more, and in particular the discrepancy between the retrieved obstacle ω and the true obstacle $\mathcal{O}$ seems hard to estimate.

3. About the method of quasi-reversibility.

3.1. **The continuous formulation.** In this paragraph, we denote $\omega \Subset D$ an open domain such that $\Omega = \mathcal{D} \setminus \overline{\omega}$ is connected. The domain ω plays the role of an updated estimate of obstacle $\mathcal{O}$. We assume that $u \in H^2(\Omega)$ solves the following ill-posed Cauchy problem in Ω:

$$\left\{ \begin{array}{rcll} \Delta u & = & 0 & \text{in } \Omega \\ u & = & g_0 & \text{on } \Gamma \\ \partial_n u & = & g_1 & \text{on } \Gamma. \end{array} \right. \tag{9}$$

Let us remark that for $N = 2, 3$ and for an obstacle $\mathcal{O}$ with Lipschitz boundary, since $u \in H^2(\Omega)$, by standard Sobolev inclusion we have $u \in C^0(\overline{\Omega})$, so that u has the regularity which is required in the previous sections. We first introduce the method of quasi-reversibility with uncontaminated data (g_0, g_1), which belong to $H^1(\Gamma) \times L^2(\Gamma)$.

We now introduce the following sets

$$V = \left\{ v \in H^2(\Omega) |\ v = g_0,\ \partial_n v = g_1 \text{ on } \Gamma \right\}$$

$$V_0 = \left\{ v \in H^2(\Omega) |\ v = 0,\ \partial_n v = 0 \text{ on } \Gamma \right\}.$$

It is clear that V_0, endowed with the classical scalar product of $H^2(\Omega)$, is a Hilbert space. In the spirit of [30, 1, 2], we introduce the following variational formulation of quasi-reversibility.

Problem [QR] : find $u_\varepsilon \in V$ such that $\forall v \in V_0$, we have

$$(\Delta u_\varepsilon, \Delta v)_{L^2(\Omega)} + \varepsilon (u_\varepsilon, v)_{H^2(\Omega)} = 0.$$

The following proposition provides the justification of the method of quasi-reversibility.

Proposition 6. *The problem [QR] has a unique solution $u_\varepsilon \in V$, such that $||u_\varepsilon - u||_{H^2(\Omega)} \to 0$ when $\varepsilon \to 0$, and we have the estimate*

$$\|\Delta u_\varepsilon - \Delta u\|_{L^2(\Omega)} \leq \sqrt{\varepsilon} \|u\|_{H^2(\Omega)}.$$

The proof of proposition 6 is very similar to the proofs provided in [1, 2] in slightly different cases, this is why it is not reproduced here.

Now we briefly describe the more delicate problem of convergence rate and the realistic case of noisy data. In order to simplify the presentation of these two problems, we assume that Ω is of class $C^{1,1}$, so that $(g_0, g_1) \in H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$.

First, the convergence result provided by proposition 6 can be complemented by the proposition 7, which directly follows from [3]. The proposition 7 underlines the logarithmic stability of the Cauchy problem for the Laplace's equation, which characterizes the strong ill-posedness of such problem.

Proposition 7. *If Ω is a $C^{1,1}$-class domain, for all $\kappa \in (0, 1)$, there exists $C > 0$ depending only on κ and Ω such that for sufficiently small $\varepsilon > 0$,*

$$\|u_\varepsilon - u\|_{H^1(\Omega)} \leq C \frac{\|u\|_{H^2(\Omega)}}{\left(\log\left[\frac{\|u\|_{H^2(\Omega)}}{\varepsilon}\right]\right)^\kappa}.$$

Remark 4. For $N = 2, 3$ and for Ω with Lipschitz boundary which satisfy the cone property with some (θ, r), an analogous result as in proposition 7 can be established, but $\kappa \in (0, 1)$ shall be replaced by $\kappa \in (0, \kappa_m)$. A minimum value of $\kappa_m \leq 1$ is specified in [4] as a function of θ and N.

Secondly, if we assume that noisy Cauchy data (g_0^δ, g_1^δ) are known instead of exact data (g_0, g_1), we have the following proposition.

Proposition 8. *If Ω is a $C^{1,1}$-class domain, and for $(g_0^\delta, g_1^\delta) \in H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$ such that*

$$||g_1^\delta - g_1||_{H^{3/2}(\Gamma)} + ||g_0^\delta - g_0||_{H^{1/2}(\Gamma)} \leq \delta,$$

the problem [QR] with (g_0^δ, g_1^δ) instead of (g_0, g_1) has a unique solution $u_\varepsilon^\delta \in V$, and there exists a constant C such that we have the estimates

$$\|u_\varepsilon^\delta - u_\varepsilon\|_{H^2(\Omega)} \leq C \frac{\delta}{\sqrt{\varepsilon}}, \quad \|\Delta u_\varepsilon^\delta - \Delta u_\varepsilon\|_{L^2(\Omega)} \leq C\,\delta.$$

The proof of proposition 8 is classical and results from Lax-Milgram theorem after using an extension $U^\delta \in H^2(\Omega)$ of $(g_0^\delta, g_1^\delta) \in H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$. Of course, the reader will easily deduce some global H^1 error estimate for $u_\varepsilon^\delta - u = (u_\varepsilon^\delta - u_\varepsilon) + (u_\varepsilon - u)$ by combining propositions 7 and 8.

A delicate problem related to noisy data concerns the choice of ε as a function of δ. In the case we know the norm c of the continuous extension operator $(g_0^\delta, g_1^\delta) \to U^\delta$, then we can modify the non-homogeneous problem [QR] into a homogeneous one with data $f^\delta = -\Delta U^\delta$, which is contaminated by some noise of known amplitude $c\delta$. This homogeneous problem coincides with a Tikhonov regularization (see [2]). Then any usual method adapted to the Tikhonov framework may be applied for choosing ε, for example the Morozov's discrepancy principle like in [2, 5] or the balancing principle like in [9]. In the general case, when the norm of the extension operator is unknown, the problem of choosing ε directly as a function of the amplitude δ of the noise that contaminates the Cauchy data (g_0, g_1) seems unsolved and should be addressed in a future paper. For that reason, in our numerical experiments, there will be no theoretical justification for such choice.

3.2. **The discretized formulation.** In view of numerical implementation in two dimensions, we introduce a discretized formulation of quasi-reversibility, precisely a finite element method. Other numerical approximations could be applied, like finite differences [6] or splines [7], but such methods are confined to simple geometries. Geometry is not a limitation for finite elements, which makes them attractive. Since the method of quasi-reversibility amounts to a fourth-order problem, Hermite finite elements are required instead of usual Lagrange finite elements. Here we use the so-called Fraeijs de Veubeke's finite element (F.V.1). This nonconforming finite element was initially introduced in [22] in order to solve plate bending problems, and its convergence was analyzed in [17]. In particular, such finite element provides a good balance between the quality of approximation and the complexity of shape functions.

We assume now that Ω is a polygonal domain in $\mathbb{R}^2$. We consider a regular triangulation $\mathcal{T}_h$ of $\overline{\Omega}$ (see [27] for definition) such that the diameter of each triangle $K \in \mathcal{T}_h$ is bounded by h. The set $\overline{\Gamma}$ consists of the union of the edges of some triangles $K \in \mathcal{T}_h$, and the complementary part of the boundary $\partial\Omega$ is denoted Γ_c.

In order to describe the F.V.1 finite element, we consider a triangle K of vertices A_i ($i = 1, 2, 3$). The indices i, j, k belong to the set $\{1, 2, 3\}$ modulo 3 in order to simplify notations. We denote M_i, the mid-point of the edge $[A_{i+1}, A_{i-1}]$, $|A_{i+1}A_{i-1}|$ its length. Lastly we denote n_i, the outward normal to the edge that is at the opposite of A_i.

The degrees of freedom for the finite element F.V.1, which are well defined for a $C^1(\overline{K})$ function w, are

- the values of the function at the vertices, namely $w(A_i)$, $i = 1, 2, 3$,
- the values at the mid-points of the edges of the element, namely $w(M_i)$, $i = 1, 2, 3$,
- the mean values of the normal derivative along each edge, namely

$$[w]_i = \frac{1}{|A_{i+1}A_{i-1}|} \int_{A_{i+1}}^{A_{i-1}} (\nabla w.n_i) \, d\Gamma, \quad i = 1, 2, 3.$$

As detailed in [17], the space of shape functions P_K in K which is associated to these degrees of freedom satisfies $P_2(K) \subset P_K \subset P_3(K)$, where $P_q(K)$ ($q = 1, 2, 3$) denotes the set of polynomials defined on K and of total degree $\leq q$.

Let W_h denote the set of functions $w_h \in L^2(\Omega)$ such that for all $K \in \mathcal{T}_h$, $w_h|_K$ belongs to the space of shape functions P_K in K, and such that the degrees of freedom coincide between two triangles that have an edge in common. Then, we define $V_{h,0}$ as the subset of functions of W_h for which the degrees of freedom on the edges contained in $\overline{\Gamma}$ vanish, and V_h as the subset of functions of W_h for which the degrees of freedom on the edges contained in $\overline{\Gamma}$ coincide with the corresponding values obtained with data g_0 and g_1 (or g_0^δ and g_1^δ in case of noisy data).

Precisely, by denoting e any edge of some triangle $K \in \mathcal{T}_h$ such that $e \subset \overline{\Gamma}$,

(10)

$$V_h = \left\{ w_h \in W_h \,|\, \forall e = [A_1, A_2] \subset \overline{\Gamma}, \begin{array}{l} w_h(A_1) = g_0(A_1), \;\; w_h(A_2) = g_0(A_2), \\ w_h(M_3) = g_0(M_3), \;\; [w_h]_3 = \dfrac{1}{|e|}\displaystyle\int_e g_1 \, d\Gamma \end{array} \right\}$$

$$V_{h,0} = \left\{ w_h \in W_h \,|\, \forall e = [A_1, A_2] \subset \overline{\Gamma}, \, w_h(A_1) = w_h(A_2) = w_h(M_3) = [w_h]_3 = 0 \right\}.$$

For some triangle K of $\mathcal{T}_h$, we denote for all functions $v, w \in H^2(K)$:

$$a_{K,\varepsilon}(v, w) = (\Delta v, \Delta w)_{L^2(K)} + \varepsilon (v, w)_{H^2(K)},$$

and we introduce the following discretized formulation of quasi-reversibility:
Problem [QRh] : find $u_{h,\varepsilon} \in V_h$ such that for all functions $v_h \in V_{h,0}$,

$$\sum_{K \in \mathcal{T}_h} a_{K,\varepsilon}(u_{h,\varepsilon}, v_h) = 0. \tag{11}$$

We begin with the following proposition.

Proposition 9. *The problem [QRh] has a unique solution $u_{h,\varepsilon}$.*

Proof. Let take any function $\tilde{u}_h$ of V_h and let us denote $w_{h,\varepsilon} = u_{h,\varepsilon} - \tilde{u}_h \in V_{h,0}$. The problem (11) is equivalent to find $w_h \in V_{h,0}$ such that for all function $v_h \in V_{h,0}$,

$$\sum_{K \in \mathcal{T}_h} a_{K,\varepsilon}(w_h, v_h) = - \sum_{K \in \mathcal{T}_h} a_{K,\varepsilon}(\tilde{u}_h, v_h).$$

This is a square system in a finite dimensional space, so uniqueness implies existence. It remains to prove uniqueness. Assume $v_h \in V_{h,0}$ satisfies

$$\sum_{K\in\mathcal{T}_h} a_{K,\epsilon}(v_h, v_h) = 0.$$

We obtain $\|v_h\|_{H^2(K)} = 0$ for all $K \in \mathcal{T}_h$, that is $v_h = 0$ in K, and hence $v_h = 0$ in Ω. □

To analyze convergence when h tends to 0, we introduce the norm $||.||_h$ in W_h, which is defined, for $w_h \in W_h$, by

$$||w_h||_h^2 = \sum_{K\in\mathcal{T}_h} ||w_h||_{H^2(K)}^2.$$

Our discretized formulation [QRh] is justified by the following convergence theorem, which in particular underlines the convergence of the discretized solution to the continuous solution when the mesh size h tends to zero, with a convergence rate proportional to h. It is proved in appendix (section 7).

Theorem 3.1. *Let u_ε denote the solution of problem [QR], and $u_{h,\varepsilon}$ the solution of problem [QRh]. We assume that $u_\varepsilon \in H^4(\Omega)$, and $\varepsilon \leq 1$. We have the error estimate:*

$$\|u_{h,\varepsilon} - u_\varepsilon\|_h \leq Ch\left(\|u_\varepsilon\|_{H^4(\Omega)} + \frac{1}{\sqrt{\varepsilon}}|u_\varepsilon|_{H^3(\Omega)} + \frac{1}{\varepsilon}\|\Delta u_\varepsilon\|_{H^2(\Omega)}\right), \tag{12}$$

where the constant C is independent of h and ε, and $||.||_{H^m(\Omega)}$ (resp. $|.|_{H^m(\Omega)}$) denotes the standard norm (resp. semi-norm) of $H^m(\Omega)$.

Remark 5. It would be interesting to obtain an error estimate directly between the solution $u_{h,\varepsilon}$ of the discretized formulation [QRh] and the exact solution u of the Cauchy problem with the help of Carleman estimates, that is without using the solution u_ε of the continuous formulation [QR], like it is done in [6] (see theorem 4.3). However, the estimate in [6] holds in a subdomain of Ω instead of in the whole domain, and in the case of a finite difference scheme instead of a finite element method.

4. **Description of our approach.** In this section, we deduce from the results of section 2 and section 3 an algorithm to approximately solve the problem (1) presented in the introduction, that is to retrieve the obstacle $\mathcal{O}$ from the Cauchy data (g_0, g_1) on Γ. We propose the following algorithm in the continuous framework.
Algorithm :

1. Choose an initial guess $\mathcal{O}_0$ such that $\mathcal{O} \subset \mathcal{O}_0 \Subset \mathcal{D}$ and $\mathcal{D} \setminus \overline{\mathcal{O}_0}$ is connected.
2. First step: the domain $\mathcal{O}_m$ being given, solve the quasi-reversibility problem [QR] in $\Omega_m := \mathcal{D} \setminus \overline{\mathcal{O}_m}$ for some selected small $\varepsilon > 0$. The solution is denoted u_m.
3. Second step: the function u_m being given, solve the non-homogeneous Dirichlet problem

$$\begin{cases} \Delta v = f & \text{in} \quad \mathcal{O}_m \\ v = |u_m| & \text{in} \quad \partial\mathcal{O}_m \end{cases} \tag{13}$$

for some selected $f \in H^{-1}(\mathcal{D})$. The solution is denoted ϕ_m. Define

$$\mathcal{O}_{m+1} = \mathcal{O}_m \setminus \text{supp}(\sup(\phi_m, 0)).$$

4. Go back to the first step until the stopping criteria is reached.

Here are some comments on the above algorithm. Concerning the initial guess, we have to choose $\mathcal{O}_0$ sufficiently big to include the unknown obstacle $\mathcal{O}$.
Concerning the first step, we need that for all m, $\mathcal{D}\setminus\overline{\mathcal{O}_m}$ be connected. This is true for example if assumptions [H1] [H2] hold (see remark 1). Furthermore, the choice of the small regularization parameter ε in problem [QR] is a tough job. This matter is discussed at the end of section 3.1, where some references are indicated, but not properly solved in the present paper. Here, we content ourselves with testing several values of ε in the next section (numerical experiments). Each time we use uncontaminated data, we choose $\varepsilon = 10^{-5}$.

Concerning the second step, it is not clear that problem (13) has always a meaning, because the regularity of the domains $\mathcal{O}_m$ is unknown. However in the case when the domains Ω_m (or equivalently $\mathcal{O}_m$) have Lipschitz boundary, then the trace of $|u_m|$ on $\partial\Omega_m = \partial\mathcal{O}_m$ is well defined in $H^{1/2}(\partial\mathcal{O}_m)$, and hence the problem (13) is uniquely solvable in $H^1(\mathcal{O}_m)$. As far as the choice of the second member f is concerned, in view of remark 2 we simply impose f to be a sufficiently large constant γ. It seems that there is no way to choose a priori the minimum value of such constant, since such minimum value depends on the exact solution u, which is unknown. The impact of that choice is discussed and a procedure of selection is suggested in the next section (numerical experiments).

Concerning the stopping criteria, several choices as possible, but in view of theorem 2.5 it is reasonable to stop the algorithm when the Hausdorff distance (for open domains) between $\mathcal{O}_m$ and $\mathcal{O}_{m+1}$ is sufficiently low. This point is also considered in the next section.

It should be noted that the boundary of the updated domain $\mathcal{O}_{m+1}$ is characterized by $\phi_m = 0$ (in the sense of trace since $\phi_m \in H^1(\mathcal{O}_m)$), that is why we can view our algorithm as an approach coupling the method of quasi-reversibility and a level set method. According to section 3, the quasi-reversibility solutions u_m are close to the exact solution u in Ω_m, and then according to section 2, our domains $\mathcal{O}_m$ have a chance to converge to a domain that is close to the exact obstacle $\mathcal{O}$.

5. **Numerical experiments.**

In this section, we present the results of numerical experiments in 2 dimensions. The domain $\mathcal{D}$ is the square $]-0.5, 0.5[\times]-0.5, 0.5[$ and we consider two non convex obstacles $\mathcal{O} \Subset \mathcal{D}$ such that $\Omega = \mathcal{D}\setminus\overline{\mathcal{O}}$ is connected.

- The first one will be named the boomerang (see the left figure of 1) and is given by the following parametric equation

$$\left\{\begin{array}{c} x(t) = 0.15\cos(t)(1+\cos(t))(1-0.5\cos(t)) - 0.1 \\ y(t) = 0.1\sin(t) - 0.2, \end{array}\right. \qquad t \in [0, 2\pi].$$

- The second one (see the right figure of 1) is the union of the disc of center $(-0.2, 0)$ and radius 0.15 and the disc of center $(0.23, 0.2)$ and radius 0.1.

Given an open subset Γ of $\partial\mathcal{D}$, our experiments are based on artificial Cauchy data $(g_0, g_1) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ which are obtained as follows. For given $\tilde{g_1} \in H^{-1/2}(\partial\mathcal{D})$, we solve the problem

$$\left\{\begin{array}{rcll} \Delta u & = & 0 & \text{in } \Omega \\ \partial_n u & = & \tilde{g_1} & \text{on } \partial\mathcal{D} \\ u & = & 0 & \text{on } \partial\mathcal{O}, \end{array}\right. \tag{14}$$

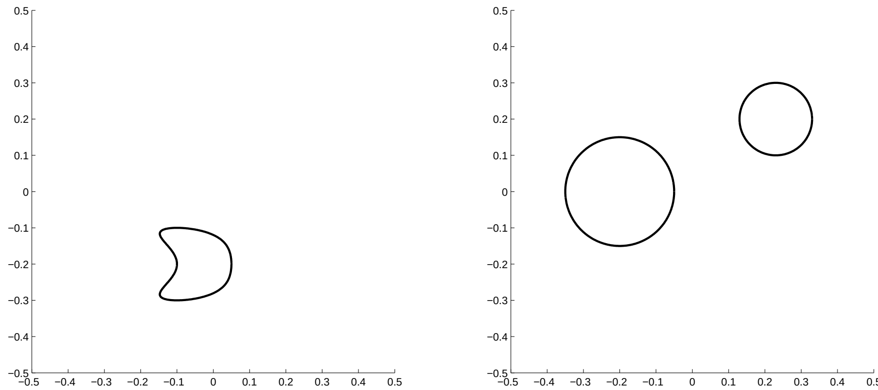

FIGURE 1. The obstacles : boomerang and two spheres

which is well-posed in $H^1(\Omega)$, and we take $(g_0, g_1) = (u|_\Gamma, \partial_n u|_\Gamma = \tilde{g}_1|_\Gamma)$. We consider a pair of Cauchy data (g_0, g_1) based on $\tilde{g}_1$ defined by

$$\left\{ \begin{array}{lll} \tilde{g}_1 = 1 & \text{on} &]-0.5, 0.5[\times\{-0.5\}\cup]-0.5, 0.5[\times\{0.5\} \\ \tilde{g}_1 = 0 & \text{on} & \{-0.5\}\times]-0.5, 0.5[\cup\{0.5\}\times]-0.5, 0.5[\end{array} \right. \tag{15}$$

in problem (14). For such $\tilde{g}_1$, since Ω is delimited by the exterior square $\mathcal{D}$ and the interior smooth obstacle $\mathcal{O}$, the solution u of (14) belongs to $H^2(\Omega)$ in virtue of [28] (see theorem 2.4.3 and remark 2.4.5), as required in the previous sections. To solve problem (14), a classical P_1 Lagrange finite element method is used, the mesh being based on a polygonal curve that approximates ∂O.

Now we indicate how the algorithm described in section 4 in the continuous framework shall be adapted to the discretized framework.

Algorithm :

1. Choose an initial guess $\mathcal{O}_{h0}$ as the union of triangles of $\mathcal{T}_h$ such that $\mathcal{O} \subset \mathcal{O}_{h0} \Subset \mathcal{D}$ and $\mathcal{D} \setminus \overline{\mathcal{O}_{h0}}$ is connected.
2. First step: the polygonal domain $\mathcal{O}_{hm}$ being given, solve the quasi-reversibility problem [QRh] in $\Omega_{hm} := \mathcal{D}\setminus\overline{\mathcal{O}_{hm}}$. The solution is denoted $u_{hm} \in V_{hm}$, where V_{hm} has the equivalent definition (10) of V_h when Ω is replaced by Ω_{hm}.
3. Second step: the function u_{hm} being given in V_{hm}, find $v_h \in V^1_{hm}$ which solves the non-homogeneous Dirichlet problem

$$\left\{ \begin{array}{c} \displaystyle\int_{\mathcal{O}_{hm}} \nabla v_h . \nabla w_h \, dx = -\int_{\mathcal{O}_{hm}} \gamma w_h \, dx, \quad \forall w_h \in V^1_{hm,0} \\ v_h - \pi_1(|u_{hm}|) \in V^1_{hm,0}, \end{array} \right. \tag{16}$$

   where V^1_{hm} denotes the space generated by standard P_1 Lagrange finite elements in the polygonal domain $\mathcal{O}_{hm}$, $V^1_{hm,0}$ the subspace of functions in V^1_{hm} that vanish on $\partial\mathcal{O}_{hm}$, and π_1 the interpolation operator on the space generated by $P1$ Lagrange finite element in the domain Ω_{hm}. The solution of (16) is denoted ϕ_{hm}. Define

$$O_{h,m+1} = \{x \in \mathcal{O}_{hm}, \quad \phi_{hm}(x) < 0\},$$

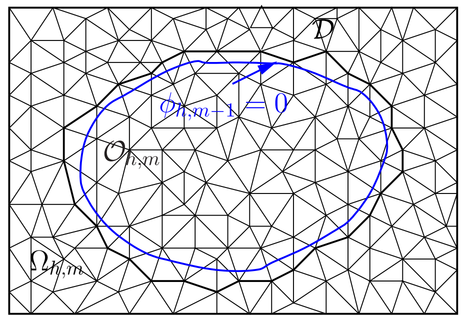

FIGURE 2. Domains Ω_{hm} and $\mathcal{O}_{hm}$

and $\mathcal{O}_{h,m+1}$ as the polygonal domain which consists of triangles of $\mathcal{T}_h$ which have at least one vertex in $O_{h,m+1}$ (see figure 2).

4. Go back to the first step until the stopping criteria is reached.

It is important to note that a single mesh is used in the algorithm, which is the same in the first and in the second step. In our numerical experiments, the domain $\mathcal{D}$ is triangulated by first dividing each edge of the square into 160 equal segments. The initial guess $\mathcal{O}_{h0}$ is delimited by a polygonal curve that approximates the circle of center $(0, 0)$ and radius 0.45.

5.1. **About the choice of constant γ.** In this section we present some results obtained for $\Gamma = \partial\mathcal{D}$ and Cauchy data (g_0, g_1) defined by (15), with various values of the constant γ in the second member of problem (16). The result of identification for the boomerang is displayed on figure 3 for $\gamma = 15$. On the left of figure 4, we compare the results of identification for $\gamma = 15$, 20 and 25 at iteration $m = 30$. On the right of figure 4, we compare the convergence rate in term of the Hausdorff distance between $\mathcal{O}_{hm}$ and $\mathcal{O}$ as a function of m, for $\gamma = 15$, 20 and 25. The

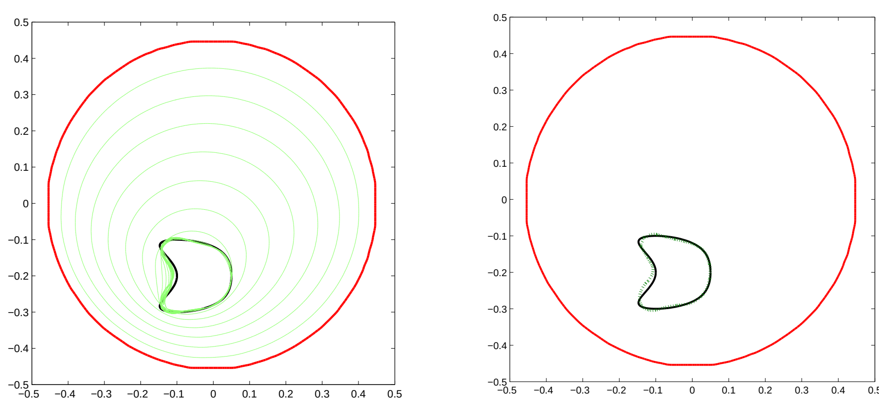

FIGURE 3. Identification of the boomerang (with and without intermediate steps): the obstacle is almost perfectly found

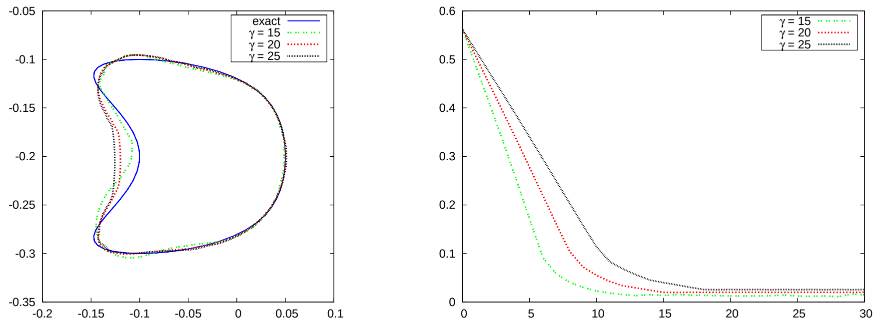

Figure 4. Impact of γ: identification (left) and convergence (right): the obstacle is almost perfectly found

result of identification for the two spheres is displayed on figure 5 for $\gamma = 5$, and a comparison of results and convergence rates is shown on figure 6 for $\gamma = 5$, 10 and 15.

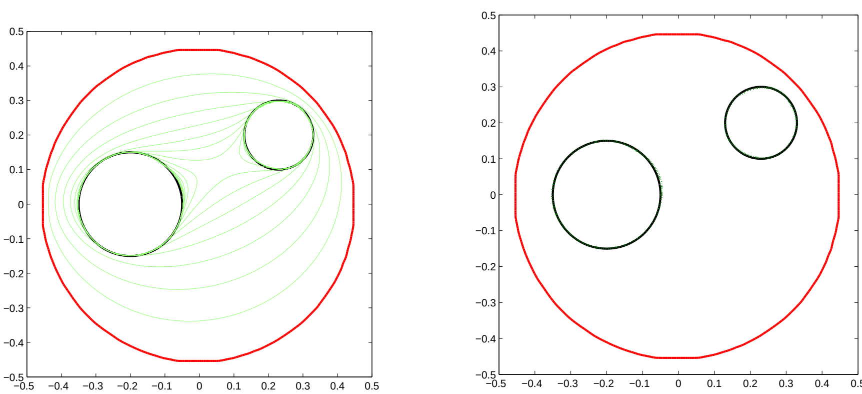

Figure 5. Identification of the two spheres (with and without intermediate steps): the obstacle is almost perfectly found

We have seen previously that our level set method is justified for sufficiently large constant γ (see remark 2). However, it can be seen on the right figures of 4 and 6 that the convergence rate decreases as γ increases: actually, when γ becomes larger, the solution ϕ_{hm} of problem (16) vanishes more rapidly towards the inside of the domain $\mathcal{O}_{hm}$. Besides, the retrieved obstacle does not depend on γ provided the number of iterations be sufficiently large. This emphasizes the need for a good stopping criteria, which is discussed in the next section. Assuming we have found such a stopping criteria, we suggest the following rule for choosing γ. We perform the method with increasing values of $\gamma > 0$, and identify the value γ_0 that stabilizes the retrieved obstacle for all $\gamma > \gamma_0$ once the stopping criteria is reached. For example, as can be seen on figure 6 for the two spheres, for $\gamma = 15$ the stopping criteria is not yet reached at iteration $m = 30$. In fact γ should not be too large

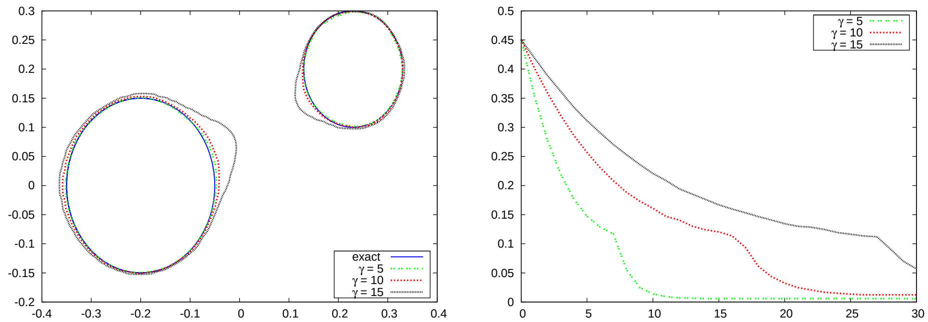

FIGURE 6. Impact of γ: identification (left) and convergence (right)

because of the discretization: actually, if γ is too large, the band within which ϕ_{hm} vanishes is narrower than the mesh size, and hence convergence stops.

5.2. **About the stopping criteria.** As previously seen, a good criteria to stop our algorithm consists in measuring the Hausdorff distance d_m between the successive open subsets $\mathcal{O}_{hm}$ and $\mathcal{O}_{h,m+1}$. As shown on figure 7, where distance d_m is plotted as a function of m for our two obstacles, d_m vanishes for sufficiently large m, which provides a robust stopping criteria.

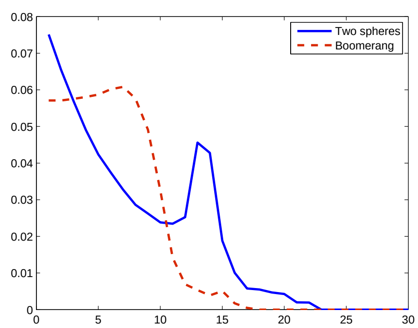

FIGURE 7. d_m versus m for the boomerang ($\gamma = 20$) and the two spheres ($\gamma = 10$)

5.3. **Some tricky Cauchy data.** We now consider in this section another pair of Cauchy data (g_0, g_1), based on $\tilde{g}_1$ defined in problem (14) by

$$\begin{cases} \tilde{g}_1 = -1 & \text{on} \quad]-0.5, 0.5[\times\{-0.5\} \\ \tilde{g}_1 = 1 & \text{on} \quad]-0.5, 0.5[\times\{0.5\} \\ \tilde{g}_1 = 0 & \text{on} \quad \{-0.5\}\times]-0.5, 0.5[\cup\{0.5\}\times]-0.5, 0.5[. \end{cases}$$

This is a tricky case because these Cauchy data are based on a solution u that vanishes not only on the boundary $\partial\mathcal{O}$ of the obstacle but also on a line that crosses $\mathcal{D}$. As shown on figure 8 for the two spheres and $\Gamma = \partial\mathcal{D}$, the identification is obviously deteriorated by such phenomenon in comparison to 5. This emphasizes the fact that from the point of view of inverse problems, certain solicitations of Neumann type, namely $\tilde{g}_1$ on $\partial\mathcal{D}$, are better than others to retrieve obstacles.

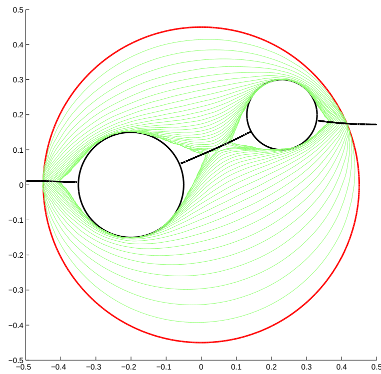

FIGURE 8. Identification with tricky Cauchy data (with intermediate steps)

5.4. **Partial data.** The use of quasi-reversibility method in our algorithm allows us to consider partial data on $\partial\mathcal{D}$, that is Γ is strictly included in $\partial\mathcal{D}$. In this section we analyze the impact of the support of Cauchy data, namely Γ, on the quality of the reconstruction. Figures 9 and 10 show the results of identification for the two spheres ($\gamma = 10$), for different Γ which are specified on the figures (Γ is represented by a bold dashed line).

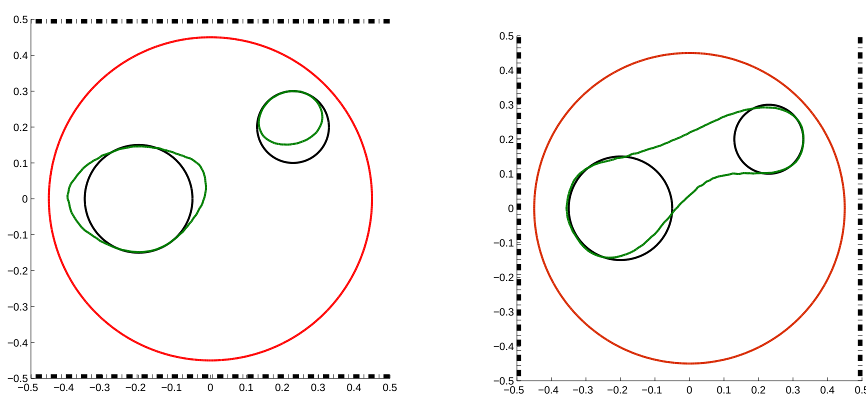

FIGURE 9. Data on two edges of the square

5.5. **About noisy data and the choice of ε.** Lastly, we analyze the impact of some noise that contaminates the Cauchy data (g_0, g_1). In this view we consider now g_0 and g_1 as vectors of components the degrees of freedom defined by (10). These components are subjected pointwise to some Gaussian noise, namely

$$(g_0^\delta, g_1^\delta) = (g_0, g_1) + \delta \frac{(||g_0||^2 + ||g_1||^2)^{\frac{1}{2}}}{(||b_0||^2 + ||b_1||^2)^{\frac{1}{2}}}(b_0, b_1),$$

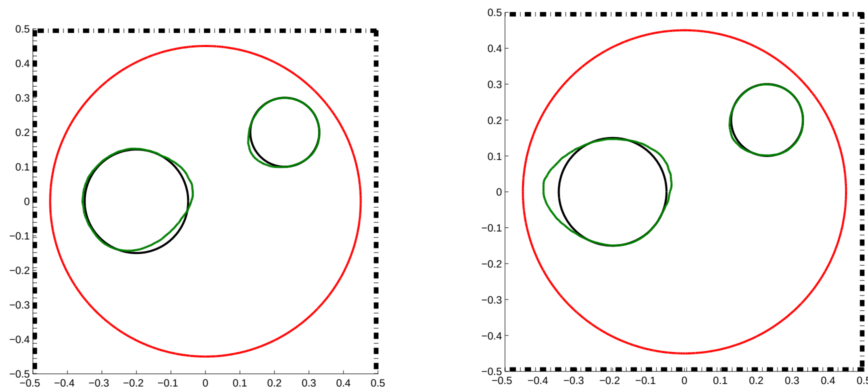

FIGURE 10. Data on three edges of the square

where (b_0, b_1) is given by a standard normal distribution, $\delta > 0$ is a scaling factor and $||.||$ denotes a discretized L^2 norm. Obviously, such definition implies that the Cauchy data (g_0, g_1) are contaminated by a relative error of amplitude δ in L^2 norm.

In figure 12, we have plotted, for our two obstacles and for three different amplitudes of noise $\delta = 0.1\%$, 0.2%, 0.5%, the Hausdorff distance between the exact and the retrieved obstacles as a function of ε, when ε ranges from 10^{-5} to 1.5×10^{-4}. Considering the "simple case" of the boomerang (figure 12, left), we observe that for small δ, the smaller is ε the better is the identification, like for $\delta = 0$ and in agreement with proposition 7. For bigger values of δ, using too small ε is inadequate and there seems to have an optimal value of ε, as suggested by proposition 8. This is confirmed by the "more complex case" of the two spheres (figure 12, right). Besides, this optimal value seems to increase when δ increases. As explained at the end of subsection 3.1, a study on a systematic way of choosing ε as a function of δ is in progress. The exact and retrieved obstacles for increasing values of noise are displayed on figure 11 for $\varepsilon = 10^{-4}$.

6. **Concluding remarks.**

We conclude our paper with a series of remarks concerning the method we have introduced to solve the inverse obstacle problem.

The main feature of our method is it does not rely on a minimization problem. This stems from the nature of the method of quasi-reversibility, which is a direct regularization method for the Cauchy problem in the sense it needs only one computation. Hence, our approach does not rely on the iterative computation of direct problems in the updated domain, as it usually happens.

Another feature is the simplicity of the level set method that we use, though it tolerates topological changes as the classical ones. The resolution of the eikonal equation is replaced by the resolution of a simple Laplace equation with constant second member. This fact implies a significant simplification in the computation of updated level set function ϕ_m, which results from a classical finite element method based on the same mesh as used for the finite element method that solves quasi-reversibility. This is in contrast to the computation of a solution of the eikonal equation, which usually results from a finite difference scheme based on a regular

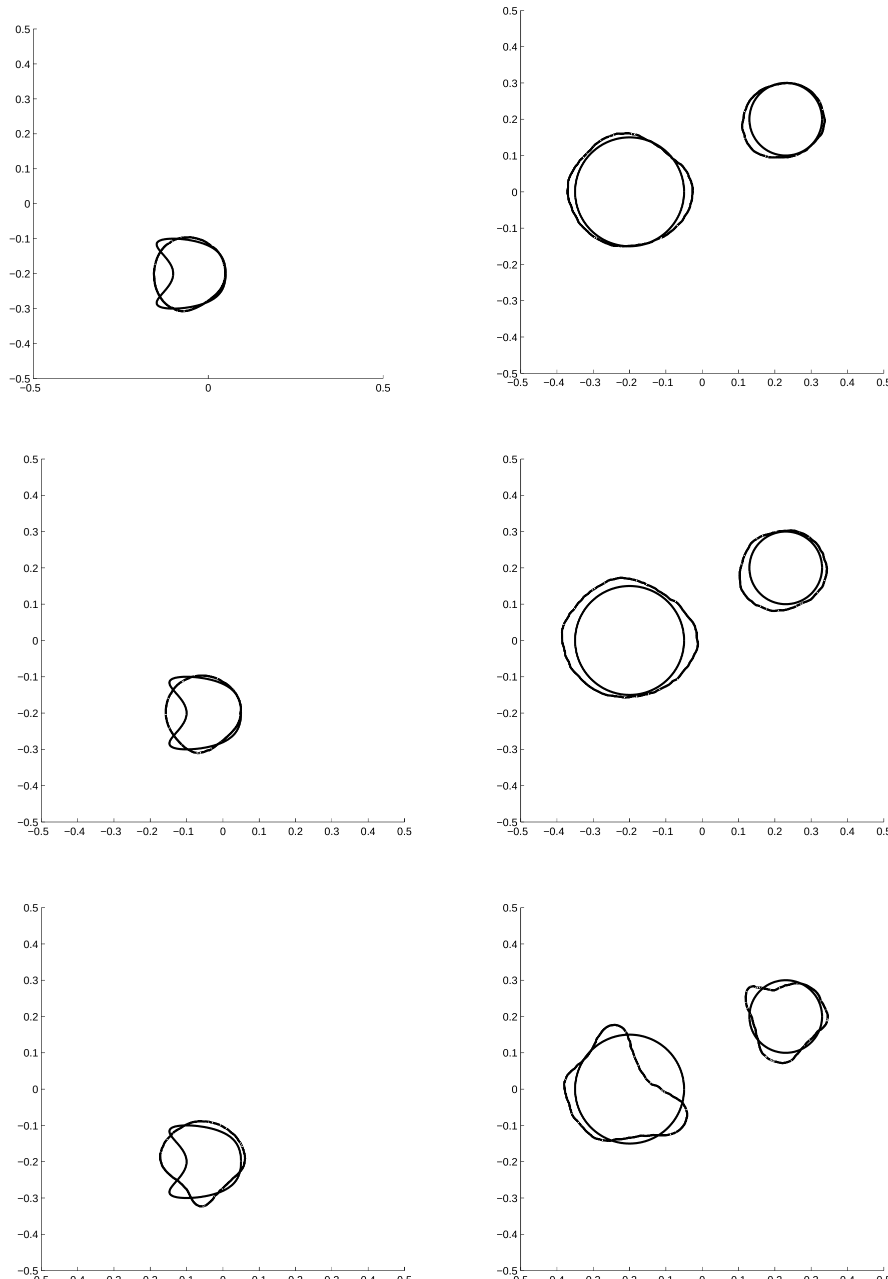

Figure 11. Exact and retrieved obstacles for $\delta = 0.1\%$ (top), 0.2% (middle), 0.5% (bottom)

grid, and which therefore requires the finite element mesh to be linked to that regular grid.

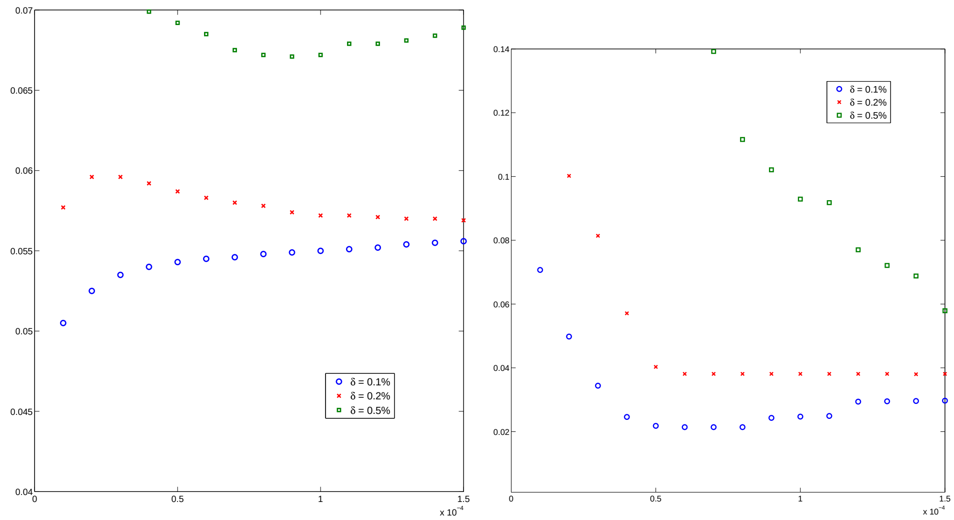

FIGURE 12. Hausdorff distance between the exact and the retrieved obstacles as a function of ε : boomerang (left) and two spheres (right)

Another significant advantage of the method of quasi-reversibility is it is applicable even if there exists a subpart of the boundary of the domain $\mathcal{D}$ on which we have no data at all. This situation cannot be easily handled by using methods that are based on the iterative computation of direct problems.
Two issues that remain partly unsolved are the *a priori* choice of the constant γ in the second member of the Laplace equation and the *a priori* choice of ε in the method of quasi-reversibility is presence of noisy data. Another challenging issue is finding an estimate of the discrepancy between the retrieved obstacle ω and the true obstacle $\mathcal{O}$ as a function of the amplitude δ of the noise contaminating (g_0, g_1), and for *ad hoc* choice of $\varepsilon(\delta)$ in the method of quasi-reversibility.

Lastly, our approach can be extended to other kinds of boundary conditions on the obstacle, provided this boundary condition depends only on the function u and its derivatives, for example $u = c$ or $|\nabla u| = c$. However, boundary conditions such as $\partial_n u = 0$ is *a priori* out of the scope of this approach, since $\partial_n u$ depends not only on u but also on n, and would require further developments.

7. **Appendix: Proof of the convergence theorem.** The aim of our appendix is to prove theorem 3.1, that is: the solution $u_{\varepsilon,h}$ of the discretized formulation of quasi-reversibility [QRh] converges to the solution u_ε of the continuous formulation of quasi-reversibility [QR], and the convergence rate is proportional to h. Our proof follows the lines of a proof used in [33], though [33] concerns the Morley's finite element and the plate bending problem. One needs the following lemma.

Lemma 7.1. *For some polygonal domain ω, for all $u \in H^4(\omega)$, for all $v \in H^2(\omega)$, one has:*

$$(\Delta u, \Delta v)_{L^2(\omega)} + \varepsilon(u,v)_{H^2(\omega)} = \int_\omega \big((1+\varepsilon)\,\Delta^2 u - \varepsilon\Delta u + \varepsilon u\big)\, v\, dx$$
$$- \int_{\partial\omega} \left((1+\varepsilon)\,\frac{\partial \Delta u}{\partial n} + \varepsilon L_2(u) - \varepsilon \frac{\partial u}{\partial n}\right) v\, d\Gamma + \int_{\partial\omega} \left((1+\varepsilon)\,\Delta u + \varepsilon L_1(u)\right) \frac{\partial v}{\partial n}\, d\Gamma.$$

Here, L_1 and L_2 are defined, by denoting (x_1,x_2) the coordinates of x, (n_1,n_2) the coordinates of the outward unit normal n, and τ the tangential vector of coordinates $(-n_2,n_1)$, by

$$L_1(u) = 2\frac{\partial^2 u}{\partial x_1 \partial x_2} n_1 n_2 - \frac{\partial^2 u}{\partial x_1^2} n_2^2 - \frac{\partial^2 u}{\partial x_2^2} n_1^2$$

$$L_2(u) = \frac{\partial}{\partial \tau}\left(\frac{\partial^2 u}{\partial x_1 \partial x_2}(n_1^2 - n_2^2) + (\frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_1^2}) n_1 n_2\right).$$

Proof. By definition of the usual scalar product in $H^2(\omega)$,

$$(u,v)_{H^2(\omega)} = \sum_{i,j=1}^{2} \int_\omega \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j} dx + \int_\omega (\nabla u.\nabla v + uv)\; dx.$$

An easy computation leads to

$$\begin{aligned}
&\sum_{i,j=1}^{2} \int_\omega \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j} dx - (\Delta u, \Delta v)_{L^2(\omega)} \\
&= \int_{\partial\omega} \frac{\partial^2 u}{\partial x_1 \partial x_2}\left[\frac{\partial v}{\partial x_2} n_1 + \frac{\partial v}{\partial x_1} n_2\right] d\Gamma \\
&- \int_{\partial\omega} \left[\frac{\partial^2 u}{\partial x_1^2}\frac{\partial v}{\partial x_2} n_2 + \frac{\partial^2 u}{\partial x_2^2}\frac{\partial v}{\partial x_1} n_1\right] d\Gamma.
\end{aligned} \tag{17}$$

We have

$$\frac{\partial v}{\partial n} = \frac{\partial v}{\partial x_1} n_1 + \frac{\partial v}{\partial x_2} n_2, \quad \frac{\partial v}{\partial \tau} = -\frac{\partial v}{\partial x_1} n_2 + \frac{\partial v}{\partial x_2} n_1,$$

whence

$$\frac{\partial v}{\partial x_1} = \frac{\partial v}{\partial n} n_1 - \frac{\partial v}{\partial \tau} n_2, \quad \frac{\partial v}{\partial x_2} = \frac{\partial v}{\partial n} n_2 + \frac{\partial v}{\partial \tau} n_1. \tag{18}$$

By plugging (18) in (17), we obtain:

$$\sum_{i,j=1}^{2} \int_\omega \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j} dx - (\Delta u, \Delta v)_{L^2(\omega)} = \int_{\partial\omega}\left(L_1(u)\frac{\partial v}{\partial n} - L_2(u) v\right) d\Gamma.$$

The proof of 7.1 results from the above identity and from the Green formula. □

Since our finite element is nonconforming, we have to pay attention to the jumps of the function and of its normal derivative across the edges of the triangles. For two triangles K_1 and K_2 of $\mathcal{T}_h$ which have a common edge e, n_{K_1} (resp. n_{K_2}) denotes the outward normal of K_1 (resp. K_2) across e. We define the normal n to e by choosing arbitrarily n_{K_1} or n_{K_2}. The jump of function $w_h \in W_h$ and of its normal derivative across e are denoted $[w_h]_e$ and $\left[\frac{\partial w_h}{\partial n}\right]_e$ and are defined as follows:

$$[w_h]_e = w_{h|K_1} n_{K_1}.n + w_{h|K_2} n_{K_2}.n, \qquad \left[\frac{\partial w_h}{\partial n}\right]_e = \frac{\partial w_{h|K_1}}{\partial n_{K_1}} + \frac{\partial w_{h|K_2}}{\partial n_{K_2}}.$$

These definitions may easily been extended to an edge e of a triangle K that belongs to Γ by denoting

$$[w_h]_e = w_{h|K}, \quad \left[\frac{\partial w_h}{\partial n}\right]_e = \frac{\partial w_{h|K}}{\partial n_K}.$$

We now prove the following proposition.

Proposition 10. *Let u_ε denote the solution of problem [QR], and $u_{h,\varepsilon}$ the solution of problem [QRh]. We assume that $u_\varepsilon \in H^4(\Omega)$, and $\varepsilon \leq 1$. We have*

$$\|u_\varepsilon - u_{h,\varepsilon}\|_h \leq \frac{1+\sqrt{3}}{\sqrt{\varepsilon}} \inf_{v_h \in V_h} \|u_\varepsilon - v_h\|_h + \frac{1}{\varepsilon} \sup_{w_h \in V_{h,0}} \frac{|F_1(w_h)|}{\|w_h\|_h} + \sup_{w_h \in V_{h,0}} \frac{|G_1(w_h)|}{\|w_h\|_h}$$
$$+ \frac{1}{\varepsilon} \sup_{w_h \in V_{h,0}} \frac{|F_2(w_h)|}{\|w_h\|_h} + \sup_{w_h \in V_{h,0}} \frac{|G_2(w_h)|}{\|w_h\|_h}, \tag{19}$$

Here, we have denoted, for $w_h \in V_{h,0}$,

$$F_1(w_h) = -\sum_{e \in S_h} \int_e \Delta u_\varepsilon \left[\frac{\partial w_h}{\partial n}\right]_e d\Gamma \tag{20}$$

$$G_1(w_h) = -\sum_{e \in S_h} \int_e (\Delta u_\varepsilon + L_1(u_\varepsilon)) \left[\frac{\partial w_h}{\partial n}\right]_e d\Gamma \tag{21}$$

$$F_2(w_h) = \sum_{e \in S_h} \int_e \frac{\partial \Delta u_\varepsilon}{\partial n} [w_h]_e \, d\Gamma \tag{22}$$

$$G_2(w_h) = \sum_{e \in S_h} \int_e \left(\frac{\partial \Delta u_\varepsilon}{\partial n} + L_2(u_\varepsilon) - \frac{\partial u_\varepsilon}{\partial n}\right) [w_h]_e \, d\Gamma, \tag{23}$$

and S_h denotes the set of all edges of the triangles $K \in \mathcal{T}_h$, excepted those which belong to $\overline{\Gamma_c}$.

Proof. First of all, we apply lemma 7.1 with $\omega = K \in \mathcal{T}_h$, $u = u_\varepsilon$ et $v = w_h$. We obtain, for all $w_h \in V_{h,0}$:

$$a_{K,\varepsilon}(u_\varepsilon, w_h) = -\int_{\partial K} I_2^\varepsilon(u_\varepsilon) w_h \, d\Gamma + \int_{\partial K} I_1^\varepsilon(u_\varepsilon) \frac{\partial w_h}{\partial n_K} \, d\Gamma,$$

with

$$I_1^\varepsilon(u) = (1+\varepsilon)\Delta u + \varepsilon L_1(u)$$

and

$$I_2^\varepsilon(u) = (1+\varepsilon)\frac{\partial \Delta u}{\partial n_K} + \varepsilon L_2(u) - \varepsilon \frac{\partial u}{\partial n_K},$$

where n_K is the outward normal to K. After summation over all triangles $K \in \mathcal{T}_h$, we obtain

$$\sum_{K \in \mathcal{T}_h} a_{K,\varepsilon}(u_\varepsilon, w_h) = -\sum_{e \in S_h} \int_e I_2^\varepsilon(u_\varepsilon)[w_h]_e \, d\Gamma + \sum_{e \in S_h} \int_e I_1^\varepsilon(u_\varepsilon) \left[\frac{\partial w_h}{\partial n}\right]_e d\Gamma.$$

Using the fact that $u_{h,\varepsilon}$ solves problem [QRh], for all $w_h \in V_{h,0}$:

$$\sum_{K \in \mathcal{T}_h} a_{K,\varepsilon}(u_{h,\varepsilon}, w_h) = 0,$$

that is

$$\sum_{K \in \mathcal{T}_h} a_{K,\varepsilon}(u_{h,\varepsilon}, w_h) = \sum_{K \in \mathcal{T}_h} a_{K,\varepsilon}(u_\varepsilon, w_h) + F_1(w_h) + \varepsilon G_1(w_h) + F_2(w_h) + \varepsilon G_2(w_h).$$

For $v_h \in V_h$, subtracting $\sum_{K\in\mathcal{T}_h} a_{K,\varepsilon}(v_h, w_h)$ to both sides of the above equality implies that for all $v_h \in V_h$, for all $w_h \in V_{h,0}$,

$$\begin{aligned}\sum_{K\in\mathcal{T}_h} a_{K,\varepsilon}(u_{h,\varepsilon} - v_h, w_h) &= \sum_{K\in\mathcal{T}_h} a_{K,\varepsilon}(u_\varepsilon - v_h, w_h) + F_1(w_h) + \varepsilon G_1(w_h) \\ &+ F_2(w_h) + \varepsilon G_2(w_h).\end{aligned}$$

We remark that for $v_h \in V_h$, $u_{h,\varepsilon} - v_h \in V_{h,0}$, which leads to

$$\begin{aligned}\sum_{K\in\mathcal{T}_h} a_{K,\varepsilon}(u_{h,\varepsilon} - v_h, u_{h,\varepsilon} - v_h) &= \sum_{K\in\mathcal{T}_h} a_{K,\varepsilon}(u_\varepsilon - v_h, u_{h,\varepsilon} - v_h) + F_1(u_{h,\varepsilon} - v_h) \\ + \varepsilon\ G_1(u_{h,\varepsilon} - v_h) &+ F_2(u_{h,\varepsilon} - v_h) + \varepsilon\ G_2(u_{h,\varepsilon} - v_h).\end{aligned}$$

We introduce the notation $\|.\|^2_{h,\varepsilon} = \sum_{K\in\mathcal{T}_h} a_{K,\varepsilon}(.,.)$. With the help of the Cauchy-Schwarz's inequality, we have for all $v_h \in V_h$:

$$\begin{aligned}\|u_{h,\varepsilon} - v_h\|^2_{h,\varepsilon} &\leq \|u_{h,\varepsilon} - v_h\|_{h,\varepsilon}\|u_\varepsilon - v_h\|_{h,\varepsilon} + |F_1(u_{h,\varepsilon} - v_h)| \\ &+ \varepsilon|G_1(u_{h,\varepsilon} - v_h)| + |F_2(u_{h,\varepsilon} - v_h)| + \varepsilon|G_2(u_{h,\varepsilon} - v_h)|,\end{aligned}$$

whence

$$\begin{aligned}\|u_{h,\varepsilon} - v_h\|_{h,\varepsilon} &\leq \|u_\varepsilon - v_h\|_{h,\varepsilon} + \frac{|F_1(u_{h,\varepsilon} - v_h)|}{\|u_{h,\varepsilon} - v_h\|_{h,\varepsilon}} \\ &+ \varepsilon\frac{|G_1(u_{h,\varepsilon} - v_h)|}{\|u_{h,\varepsilon} - v_h\|_{h,\varepsilon}} + \frac{|F_2(u_{h,\varepsilon} - v_h)|}{\|u_{h,\varepsilon} - v_h\|_{h,\varepsilon}} + \varepsilon\frac{|G_2(u_{h,\varepsilon} - v_h)|}{\|u_{h,\varepsilon} - v_h\|_{h,\varepsilon}}.\end{aligned}$$

Using the two inequalities

$$\|w\|^2_{h,\varepsilon} = \sum_{K\in\mathcal{T}_h}\left[\|\Delta w\|^2_{L^2(K)} + \varepsilon\|w\|^2_{H^2(K)}\right] \geq \varepsilon\|w\|^2_h$$

and

$$\|w\|^2_{h,\varepsilon} \leq \sum_{K\in\mathcal{T}_h}\left[2\|w\|^2_{H^2(K)} + \varepsilon\|w\|^2_{H^2(K)}\right] \leq (2+\varepsilon)\|w\|^2_h,$$

we obtain that for all $v_h \in V_h$:

$$\begin{aligned}\sqrt{\varepsilon}\|u_{h,\varepsilon} - v_h\|_h &\leq \sqrt{2+\varepsilon}\|u_\varepsilon - v_h\|_h + \frac{1}{\sqrt{\varepsilon}}\frac{|F_1(u_{h,\varepsilon} - v_h)|}{\|u_{h,\varepsilon} - v_h\|_h} + \sqrt{\varepsilon}\frac{|G_1(u_{h,\varepsilon} - v_h)|}{\|u_{h,\varepsilon} - v_h\|_h} \\ &+ \frac{1}{\sqrt{\varepsilon}}\frac{|F_2(u_{h,\varepsilon} - v_h)|}{\|u_{h,\varepsilon} - v_h\|_h} + \sqrt{\varepsilon}\frac{|G_2(u_{h,\varepsilon} - v_h)|}{\|u_{h,\varepsilon} - v_h\|_h} \\ &\leq \sqrt{3}\|u_\varepsilon - v_h\|_h + \frac{1}{\sqrt{\varepsilon}}\sup_{w_h\in V_{h,0}}\frac{|F_1(w_h)|}{\|w_h\|_h} + \sqrt{\varepsilon}\sup_{w_h\in V_{h,0}}\frac{|G_1(w_h)|}{\|w_h\|_h} \\ &+ \frac{1}{\sqrt{\varepsilon}}\sup_{w_h\in V_{h,0}}\frac{|F_2(w_h)|}{\|w_h\|_h} + \sqrt{\varepsilon}\sup_{w_h\in V_{h,0}}\frac{|G_2(w_h)|}{\|w_h\|_h},\end{aligned}$$

and then

$$\begin{aligned}\|u_{h,\varepsilon} - v_h\|_h &\leq \sqrt{\frac{3}{\varepsilon}}\|u_\varepsilon - v_h\|_h + \frac{1}{\varepsilon}\sup_{w_h\in V_{h,0}}\frac{|F_1(w_h)|}{\|w_h\|_h} + \sup_{w_h\in V_{h,0}}\frac{|G_1(w_h)|}{\|w_h\|_h} \\ &+ \frac{1}{\varepsilon}\sup_{w_h\in V_{h,0}}\frac{|F_2(w_h)|}{\|w_h\|_h} + \sup_{w_h\in V_{h,0}}\frac{|G_2(w_h)|}{\|w_h\|_h}.\end{aligned}$$

It remains to use the fact that $\|u_\varepsilon - u_{h,\varepsilon}\|_h \leq \|u_\varepsilon - v_h\|_h + \|u_{h,\varepsilon} - v_h\|_h$ and to take the inf in $v_h \in V_h$ to obtain (19). □

In order to complete our proof of convergence, we need the following lemma, which is proved in [27] (theorem 4.2.5).

Lemma 7.2. *For some polygonal domain ω, let k and l be two integers and U an Hilbert space satisfying $P_l(\omega) \subset U \subset H^{l+1}(\omega)$ (U is equipped with the norm $\|.\|_{H^{l+1}(\omega)}$). We assume that $B : H^{k+1}(\omega) \times U \to \mathbb{R}$ is a continuous bilinear form satisfying*

$$B(u,v) = 0, \ \forall u \in P_k(\omega), \ \forall v \in U,$$
$$B(u,v) = 0, \ \forall u \in H^{k+1}(\omega), \ \forall v \in P_l(\omega).$$

Then there exists c which depends only on ω such that for all $u \in H^{k+1}(\omega)$ and $v \in U$

$$|B(u,v)| \leq c\|B\||u|_{H^{k+1}(\omega)}|v|_{H^{l+1}(\omega)}$$

where $|.|_{H^m(\omega)}$ denotes the standard semi-norm of $H^m(\omega)$.

We are now in a position to give the proof of the convergence theorem 3.1.

Proof. We first consider the term $\inf_{v_h \in V_h} \|u_\varepsilon - v_h\|_h$ in (19). By setting $v_h = \pi_h(u_\varepsilon)$, where $\pi_h(u_\varepsilon)$ is the interpolate of u_ε in V_h, we directly use the interpolation result given in [17] to obtain

$$\inf_{v_h \in V_h} \|u_\varepsilon - v_h\|_h \leq ch|u_\varepsilon|_{H^3(\Omega)}.$$

Now consider the term $\sup_{w_h \in V_{h,0}} \frac{|F_1(w_h)|}{\|w_h\|_h}$, with for $w_h \in V_{h,0}$,

$$F_1(w_h) = -\sum_{e \in S_h} \int_e \Delta u_\varepsilon \left[\frac{\partial w_h}{\partial n}\right]_e d\Gamma.$$

We consider the operator π_0 as follows:

$$\begin{array}{cccc} \pi_0 : & L^2(e) & \longrightarrow & \mathbb{R} \\ & g & \longmapsto & \frac{1}{|e|}\int_e g \, d\Gamma. \end{array}$$

By definition of the finite element F.V.1, we have for all $w_h \in V_{h,0}$, $\pi_0([\partial_n w_h]_e) = 0$, and we remark that

$$\pi_0(\left[\frac{\partial w_h}{\partial n}\right]_e) = \left[\pi_0(\frac{\partial w_h}{\partial n})\right]_e,$$

which leads to

$$\int_e \left[\frac{\partial w_h}{\partial n} - \pi_0(\frac{\partial w_h}{\partial n})\right]_e d\Gamma = 0.$$

We conclude that for all $e \in S_h$, for all $w_h \in V_{h,0}$,

$$\int_e \Delta u_\varepsilon \left[\frac{\partial w_h}{\partial n}\right]_e d\Gamma = \int_e (\Delta u_\varepsilon - \pi_0(\Delta u_\varepsilon)) \left[\frac{\partial w_h}{\partial n} - \pi_0(\frac{\partial w_h}{\partial n})\right]_e d\Gamma.$$

Let F_K denote the affine transformation which maps the reference triangle $\widehat{K}$ to K, $\widehat{e} = F_K^{-1}(e)$, and $\widehat{v} = v \circ F_K$ for any function v defined on K (for all details concerning the affine theory, see [27]). We now consider the bilinear form on $H^1(\widehat{K}) \times P_2(\widehat{K})$ defined by:

$$B(\widehat{u}, \widehat{p}) = \int_{\widehat{e}} (\widehat{u} - \widehat{\pi}_0(\widehat{u}))(\widehat{p} - \widehat{\pi}_0(\widehat{p})) \, d\widehat{\Gamma}.$$

B satisfies the assumptions of lemma 7.2 with $\omega = \widehat{K}$, $U = P_2(\widehat{K})$, and $k = l = 0$. Hence there exists a constant $\widehat{c}$ such that

$$B(\widehat{u}, \widehat{v}) \leq \widehat{c}|\widehat{u}|_{H_1(\widehat{K})}|\widehat{p}|_{H^1(\widehat{K})}.$$

Let K_1 and K_2 be two triangles sharing the edge e. Going back to the reference triangle $\widehat{K}$, we obtain a constant c such that

$$\Big| \int_e \Delta u_\varepsilon \left[\frac{\partial w_h}{\partial n}\right]_e d\Gamma\Big| \leq ch(|\Delta u_\varepsilon|_{H^1(K_1)}|w_h|_{H^2(K_1)} + |\Delta u_\varepsilon|_{H^1(K_2)}|w_h|_{H^2(K_2)}).$$

If we now consider an edge $e \in \Gamma$, and $K \subset \mathcal{T}_h$ the triangle which contains it, we obtain similarly

$$\Big| \int_e \Delta u_\varepsilon \left[\frac{\partial w_h}{\partial n}\right]_e d\Gamma\Big| \leq ch|\Delta u_\varepsilon|_{H^1(K)}|w_h|_{H^2(K)}.$$

We hence obtain that for all $w_h \in V_{h,0}$:

$$\begin{aligned} |F_1(w_h)| &\leq ch \sum_{e\in S_h,\, e\in\Gamma} |\Delta u_\varepsilon|_{H^1(K)}|w_h|_{H^2(K)} + \\ &\quad ch \sum_{e\in S_h,\, e\notin\Gamma} (|\Delta u_\varepsilon|_{H^1(K_1)}|w_h|_{H^2(K_1)} + |\Delta u_\varepsilon|_{H^1(K_2)}|w_h|_{H^2(K_2)}) \\ &\leq 3ch \sum_{K\in\mathcal{T}_h} |\Delta u_\varepsilon|_{H^1(K)}|w_h|_{H^2(K)}. \end{aligned}$$

By using Cauchy-Schwarz's inequality, it follows that

$$|F_1(w_h)| \leq 3Ch|\Delta u_\varepsilon|_{H1(\Omega)}\|w_h\|_h,$$

which leads to

$$\sup_{w_h\in V_{h,0}} \frac{|F_1(w_h)|}{\|w_h\|_h} \leq Ch|\Delta u_\varepsilon|_{H^1(\Omega)},$$

with a constant C which depends neither on h, nor on ε. We prove exactly the same way that

$$\sup_{w_h\in V_{h,0}} \frac{|G_1(w_h)|}{\|w_h\|_h} \leq C'h|u_\varepsilon|_{H^3(\Omega)}.$$

Let us now denote $f = \partial_n \Delta u_\varepsilon$. We have for all $w_h \in V_{h,0}$,

$$F_2(w_h) = \sum_{e\in S_h} \int_e f[w_h]_e \, d\Gamma.$$

We define, for $K \in \mathcal{T}_h$, the operator π_1 as follows:

$$\begin{array}{rccc} \pi_1 : & P_K & \longrightarrow & P_1(K) \\ & p & \longmapsto & \sum_{i=1}^3 p(A_i)\lambda_i, \end{array}$$

where the $\lambda_i \in P_1(K)$ are uniquely defined by $\lambda_i(A_j) = \delta_{ij},\ i,j = 1,2,3$. By definition of the finite element F.V.1, for some $e \in S_h$, $[\pi_1(w_h)]_e = 0$, whence

$$\int_e f\,[w_h]_e \; d\Gamma = \int_e f\,[w_h - \pi_1(w_h)]_e \; d\Gamma.$$

Going back to the reference triangle $\widehat{K}$, we consider the following bilinear form

$$\tilde{B}(\widehat{f}, \widehat{w_h}) = \int_{\widehat{e}} \widehat{f}(\widehat{w_h} - \widehat{\pi}_1(\widehat{w_h}))d\widehat{\Gamma}.$$

By a trace inequality and a classical error interpolation on P_1, it follows that

$$\tilde{B}(\widehat{f}, \widehat{w_h}) \leq \widehat{c}\|\widehat{f}\|_{H_1(\widehat{K})}\|\widehat{w_h} - \widehat{\pi}_1(\widehat{w_h})\|_{H^1(\widehat{K})} \leq \widehat{C}\|\widehat{f}\|_{H^1(\widehat{K})}|\widehat{w_h}|_{H^2(\widehat{K})}.$$

For $e = K_1 \cap K_2$, with $K_i \in \mathcal{T}_h$, there exists a constant c which depends neither on h, nor on ε, such that

$$\int_e f\,[w_h]_e\; d\Gamma \leq ch(\|f\|_{H^1(K_1)}|w_h|_{H^2(K_1)} + \|f\|_{H^1(K_2)}|w_h|_{H^2(K_2)}).$$

For $e \in \overline{\Gamma}$, we have

$$\int_e f\,[w_h]_e\; d\Gamma \leq ch\|f\|_{H^1(K)}|w_h|_{H^2(K)}.$$

We conclude that

$$\sup_{w_h \in V_{h,0}} \frac{|F_2(w_h)|}{\|w_h\|_h} \leq Ch\|\Delta u_\varepsilon\|_{H^2(\Omega)}$$

and prove exactly the same way that

$$\sup_{w_h \in V_{h,0}} \frac{|G_2(w_h)|}{\|w_h\|_h} \leq Ch\|u_\varepsilon\|_{H^4(\Omega)}.$$

The estimate (12) follows. □

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