Stability estimates for Navier-Stokes equations and application to inverse problems

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Abstract

We first present some new Carleman inequalities for Stokes and Oseen equations with non-homogeneous boundary conditions. These estimates lead to log type stability inequalities for the problem of recovering the solution of the Stokes and Navier-Stokes equations from both boundary and distributed observations. These inequalities fit the well-known unique continuation result of Fabre and Lebeau [17]: the distributed observation only depends on interior measurement of the velocity, and the boundary observation only depends on the trace of the velocity and of the Cauchy stress tensor measurements. Finally, we present two applications for such inequalities. First, we apply these estimates to obtain stability inequalities for the inverse problem of recovering Navier or Robin boundary coefficients from boundary measurements. Next, we use these estimates to deduce the rate of convergence of two reconstruction methods of the Stokes solution from the measurement of Cauchy data: a quasi-reversibility method and a penalized Kohn-Vogelius method.

Keywords: Stability estimate, Navier-Stokes equations, Carleman inequality, Inverse problems

AMS Classification: 35R30, 35Q30, 76D07, 76D05

1 Introduction and main results

For a nonempty bounded open subset Ω of $\mathbb{R}^N$ ($N = 2$ or $N = 3$), we consider a pair velocity-pressure $(\boldsymbol{v}, p) \in \mathbf{H}^2(\Omega) \times \mathrm{H}^1(\Omega)$ solution of the following linearized Navier-Stokes equations (also called Oseen equations):

$$\left\{ \begin{array}{rcll} -\nu\Delta \boldsymbol{v} + (\boldsymbol{z}_1 \cdot \nabla)\, \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\, \boldsymbol{z}_2 + \nabla p & = & \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{v} & = & d & \text{in } \Omega. \end{array} \right. \tag{1.1}$$

Above and in the following, $\nu > 0$ is a constant which represents the kinematic viscosity of the fluid, $\boldsymbol{f} \in \mathbf{L}^2(\Omega)$, $d \in \mathrm{H}^1(\Omega)$ and

$$\boldsymbol{z}_1 \in \mathbf{L}^\infty(\Omega) \quad \text{and} \quad \boldsymbol{z}_2 \in \mathbf{W}^{1,r}(\Omega) \;\text{ with } \left\{ \begin{array}{ll} r > 2 & \text{if } N = 2, \\ r = 3 & \text{if } N = 3. \end{array} \right. \tag{1.2}$$

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In the following, $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$ will be two solutions of the Navier-Stokes equations in Ω. More precisely, if $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$ are two solutions of the Navier-Stokes equations, then their difference $\boldsymbol{v} = \boldsymbol{z}_1 - \boldsymbol{z}_2$ verifies (1.1).

System (1.1) is not completely determined since there is no condition imposed on the boundary $\partial\Omega$ of Ω. However, if we have some additional *observation*, such as the value of the velocity $\boldsymbol{v}$ in a nonempty (and arbitrary small) open subset $\omega \subset \Omega$ or the value of the Cauchy data $(\boldsymbol{v}, \sigma(\boldsymbol{v}, p)\mathbf{n})$ on a nonempty open subset Γ_{obs} of $\partial\Omega$, then Fabre and Lebeau's Theorem guarantees the uniqueness of the corresponding pair $(\boldsymbol{v}, p)$ (see [17]). However, the related stability inequality expressing the (conditional) continuous dependence of $(\boldsymbol{v}, p)$ with respect to $\|\boldsymbol{f}\|_{\mathbf{L}^2(\Omega)}$, $\|d\|_{\mathrm{H}^1(\Omega)}$ and to some norm $\|(\boldsymbol{v}, p)\|_{\text{Obs}}$ (corresponding to one of the above mentioned observation) are not yet proved for system (1.1).

The first main results of the present paper are stability inequalities for the Oseen equations (1.1) which are quantified versions of Fabre and Lebeau's uniqueness Theorem. It allows to obtain analogous stability inequalities for the Navier-Stokes equations. Then, in a second step, we give examples of applications for some parameter identification problems as well as for some error estimates for numerical reconstruction methods.

Stability inequalities. In order to state our main theorem, we need some assumptions and notations. Here and in the following, $C > 0$ denotes a generic constant which only depends on the geometry and which may change from line to line, and $K \geq e^e$ denotes a constant which satisfies:

$$\max\left\{1\,,\ \|\boldsymbol{z}_1\|_{\mathbf{L}^\infty(\Omega)}\ ,\ \|\nabla\boldsymbol{z}_2\|_{\mathbf{L}^r(\Omega)}\right\} \leq \ln(\ln K). \tag{1.3}$$

Moreover, ω denotes a nonempty open subset of Ω and Γ_{obs} denotes a nonempty open subset of $\partial\Omega$. In this paper, $\mathbf{n}$ is the outward unit normal to $\partial\Omega$ which is assumed to be of class C^2 and the stress tensor is defined by $\sigma(\boldsymbol{u}, p) \overset{\text{def}}{=} 2\nu\mathcal{D}(\boldsymbol{u}) - p\,\mathrm{I}$, where I is the identity matrix and $\mathcal{D}(\boldsymbol{y}) \overset{\text{def}}{=} \frac{1}{2}\left(\nabla\boldsymbol{y} + {}^t\nabla\boldsymbol{y}\right)$ is the symmetrized gradient.

We prove (see Subsections 3.1 and 3.2) the following

Theorem 1.1. *Assume* (1.2) *and* (1.3) *and that* $(\boldsymbol{v}, p) \in \mathbf{H}^2(\Omega) \times \mathrm{H}^1(\Omega)$ *is a solution of the Oseen equations* (1.1)*. Then the following estimates hold:*

$$\|\boldsymbol{v}\|_{\mathbf{L}^2(\Omega)} \leq \frac{CK\left(\|\boldsymbol{v}\|_{\mathbf{H}^2(\Omega)} + \|p\|_{\mathrm{H}^1(\Omega)}\right)}{\ln\left(1 + \dfrac{\|\boldsymbol{v}\|_{\mathbf{H}^2(\Omega)} + \|p\|_{\mathrm{H}^1(\Omega)}}{\|\boldsymbol{f}\|_{\mathbf{L}^2(\Omega)} + \|d\|_{\mathrm{H}^1(\Omega)} + \|\boldsymbol{v}\|_{\mathbf{L}^2(\omega)}}\right)} \tag{1.4}$$

and

$$\|\boldsymbol{v}\|_{\mathbf{L}^2(\Omega)} \leq \frac{CK\left(\|\boldsymbol{v}\|_{\mathbf{H}^2(\Omega)} + \|p\|_{\mathrm{H}^1(\Omega)}\right)}{\ln\left(1 + \dfrac{\|\boldsymbol{v}\|_{\mathbf{H}^2(\Omega)} + \|p\|_{\mathrm{H}^1(\Omega)}}{\|\boldsymbol{f}\|_{\mathbf{L}^2(\Omega)} + \|d\|_{\mathrm{H}^1(\Omega)} + \|\boldsymbol{v}\|_{\mathbf{H}^{3/2}(\Gamma_{\text{obs}})} + \|\sigma(\boldsymbol{v}, p)\mathbf{n}\|_{\mathbf{H}^{1/2}(\Gamma_{\text{obs}})}}\right)}. \tag{1.5}$$

Moreover, we have

$$\begin{aligned} &\|\operatorname{curl}\boldsymbol{v}\|_{(\mathrm{L}^2(\Omega))^{2N-3}} + \|p - \operatorname{div}\boldsymbol{v}\|_{\mathrm{L}^2(\Omega)} \\ &\leq \frac{CK\left(\|\boldsymbol{v}\|_{\mathbf{H}^2(\Omega)} + \|p\|_{\mathrm{H}^1(\Omega)}\right)}{\left(\ln\left(1 + \dfrac{\|\boldsymbol{v}\|_{\mathbf{H}^2(\Omega)} + \|p\|_{\mathrm{H}^1(\Omega)}}{\|\boldsymbol{f}\|_{\mathbf{L}^2(\Omega)} + \|d\|_{\mathrm{H}^1(\Omega)} + \|\boldsymbol{v}\|_{\mathbf{H}^{3/2}(\Gamma_{\text{obs}})} + \|\sigma(\boldsymbol{v}, p)\mathbf{n}\|_{\mathbf{H}^{1/2}(\Gamma_{\text{obs}})}}\right)\right)^{1/2}}. \end{aligned} \tag{1.6}$$

The above theorem allows us to obtain stability estimates for the Navier-Stokes equations. Let $(\boldsymbol{z}_i,\pi_i)\in\mathbf{H}^2(\Omega)\times\mathrm{H}^1(\Omega)$, $i=1,2$, satisfy

$$\left\{\begin{array}{rcll} -\nu\Delta\boldsymbol{z}_i+(\boldsymbol{z}_i\cdot\nabla)\,\boldsymbol{z}_i+\nabla\pi_i &=& \boldsymbol{f} & \text{in }\Omega,\\ \operatorname{div}\boldsymbol{z}_i &=& d & \text{in }\Omega. \end{array}\right. \tag{1.7}$$

Note that the $\mathbf{H}^2$ regularity of $\boldsymbol{z}_1$, $\boldsymbol{z}_2$ implies (1.2). We prove (see Subsection 3.3) the following

Theorem 1.2. *Suppose that* $(\boldsymbol{z}_i,\pi_i)\in\mathbf{H}^2(\Omega)\times\mathrm{H}^1(\Omega)$, $i=1,2$, *are two solutions of* (1.7) *which satisfy* (1.3) *for some* $K>e^{e}$. *Then the following estimates hold:*

$$\|\boldsymbol{z}_1-\boldsymbol{z}_2\|_{\mathbf{L}^2(\Omega)}\leq\frac{CK\left(\|\boldsymbol{z}_1-\boldsymbol{z}_2\|_{\mathbf{H}^2(\Omega)}+\|\pi_1-\pi_2\|_{\mathrm{H}^1(\Omega)}\right)}{\ln\left(1+\dfrac{\|\boldsymbol{z}_1-\boldsymbol{z}_2\|_{\mathbf{H}^2(\Omega)}+\|\pi_1-\pi_2\|_{\mathrm{H}^1(\Omega)}}{\|\boldsymbol{z}_1-\boldsymbol{z}_2\|_{\mathbf{L}^2(\omega)}}\right)} \tag{1.8}$$

and

$$\|\boldsymbol{z}_1-\boldsymbol{z}_2\|_{\mathbf{L}^2(\Omega)}\leq\frac{CK(\|\boldsymbol{z}_1-\boldsymbol{z}_2\|_{\mathbf{H}^2(\Omega)}+\|\pi_1-\pi_2\|_{\mathrm{H}^1(\Omega)})}{\ln\left(1+\dfrac{\|\boldsymbol{z}_1-\boldsymbol{z}_2\|_{\mathbf{H}^2(\Omega)}+\|\pi_1-\pi_2\|_{\mathrm{H}^1(\Omega)}}{\|\boldsymbol{z}_1-\boldsymbol{z}_2\|_{\mathbf{H}^{3/2}(\Gamma_{\mathrm{obs}})}+\|\sigma(\boldsymbol{z}_1,\pi_1)\mathbf{n}-\sigma(\boldsymbol{z}_2,\pi_2)\mathbf{n}\|_{\mathbf{H}^{1/2}(\Gamma_{\mathrm{obs}})}}\right)}. \tag{1.9}$$

Moreover, we have

$$\begin{aligned}&\|\operatorname{curl}(\boldsymbol{z}_1-\boldsymbol{z}_2)\|_{(\mathrm{L}^2(\Omega))^{2N-3}}+\|\pi_1-\pi_2\|_{\mathrm{L}^2(\Omega)}\\ &\quad\leq\frac{CK(\|\boldsymbol{z}_1-\boldsymbol{z}_2\|_{\mathbf{H}^2(\Omega)}+\|\pi_1-\pi_2\|_{\mathrm{H}^1(\Omega)})}{\ln\left(1+\dfrac{\|\boldsymbol{z}_1-\boldsymbol{z}_2\|_{\mathbf{H}^2(\Omega)}+\|\pi_1-\pi_2\|_{\mathrm{H}^1(\Omega)}}{\|\boldsymbol{z}_1-\boldsymbol{z}_2\|_{\mathbf{H}^{3/2}(\Gamma_{\mathrm{obs}})}+\|\sigma(\boldsymbol{z}_1,\pi_1)\mathbf{n}-\sigma(\boldsymbol{z}_2,\pi_2)\mathbf{n}\|_{\mathbf{H}^{1/2}(\Gamma_{\mathrm{obs}})}}\right)^{1/2}}.\end{aligned} \tag{1.10}$$

We stress that these stability estimates respect the well known unique continuation result of Fabre and Lebeau (see [17]) since the observation in ω only concerns the velocity, and since the observation on Γ_{obs} only concerns $\boldsymbol{v}_{|\Gamma_{\mathrm{obs}}}$ and $\sigma(\boldsymbol{v},p)\mathbf{n}|_{\Gamma_{\mathrm{obs}}}$. Indeed, Fabre and Lebeau's Theorem states that every velocity $\boldsymbol{v}$ solution of

$$\left\{\begin{array}{rcll} -\Delta\boldsymbol{v}+\nabla p &=& \mathbf{0} & \text{in }\ \Omega,\\ \operatorname{div}\boldsymbol{v} &=& 0 & \text{in }\ \Omega, \end{array}\right. \tag{1.11}$$

which is identically zero in ω must be zero in Ω (and then p is constant, see [17, Proposition 1.1] for precise statements). In particular, no information is required on p to obtain this result. Moreover, as a direct consequence of the above mentioned uniqueness result, we can easily deduce that, if a smooth solution $(\boldsymbol{v},p)$ of System (1.11) satisfies $\boldsymbol{v}=\mathbf{0}$ and $\sigma(\boldsymbol{v},p)\mathbf{n}=\mathbf{0}$ on Γ_{obs}, then, $\boldsymbol{v}=\mathbf{0}$ and $p=0$ in Ω. Therefore, inequalities (1.4), (1.5) and (1.6) are quantifications of Fabre and Lebeau's uniqueness theorem.

In this sense, Theorem 1.1 improves the recent work of Boulakia *et al.* (see [9]). Indeed, as explained by the authors themselves, the results stated in [9, Theorem 1.4] do not fit Fabre and Lebeau's Theorem: in the case of distributed observation the solution $(\boldsymbol{v},p)$ of the Stokes system (1.11) in the whole domain Ω is estimated with respect to the value of both $\boldsymbol{v}$ and p in ω, and in the case of boundary observation, the solution is estimated with respect to $\boldsymbol{v}_{|\Gamma_{\mathrm{obs}}}$, $p_{|\Gamma_{\mathrm{obs}}}$, $\partial_{\mathbf{n}}\boldsymbol{v}_{|\Gamma_{\mathrm{obs}}}$ and also $\partial_{\mathbf{n}}p_{|\Gamma_{\mathrm{obs}}}$.

The proof of Theorem 1.1 is based on global Carleman inequalities for the Oseen system with non-homogeneous data. Quantitative results for unique continuation are classically obtained thanks to Carleman inequalities and three-spheres inequalities. We refer to the topical review of Alessandrini *et al.* [3] and to the references therein for elliptic cases; see also the works of Le Rousseau *et al.* in [24]. However, there is not so much results available on quantitative uniqueness for systems. About Stokes system we shall mention the works of Boulakia *et al.* in [8, 9] for stability estimates and of Ballerini in [5] and Lin *et al.* in [25] for some other connected results.

Applications to inverse problems. We obtain stability inequalities for the problem of recovering Navier or Robin boundary coefficients. For this, we assume that Γ_{obs} and Γ_0 are two nonempty open subsets of $\partial\Omega$ such that $\Gamma_{\text{obs}} \cap \Gamma_0 = \emptyset$ and we consider on Γ_0 a non penetration condition given by $\boldsymbol{z} \cdot \mathbf{n} = 0$ and a friction law given by $2\nu \left[\mathcal{D}(\boldsymbol{z})\mathbf{n}\right]_\tau + \alpha \boldsymbol{z} = \mathbf{0}$ (subscript τ denotes the tangential component). The aim is to reconstruct the friction coefficient α from Cauchy data on Γ_{obs}. Thus, we consider two solutions $(\boldsymbol{z}_i, \pi_i) \in \mathbf{H}^2(\Omega) \times \mathrm{H}^1(\Omega)$ $(i = 1, 2)$ of the Navier-Stokes equations

$$\left\{ \begin{array}{rcll} -\nu\Delta \boldsymbol{z}_i + (\boldsymbol{z}_i \cdot \nabla)\, \boldsymbol{z}_i + \nabla \pi_i & = & \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{z}_i & = & d & \text{in } \Omega, \end{array} \right. \tag{1.12}$$

associated to two friction coefficients $\alpha_i \in \mathbf{H}^{3/2}(\Gamma_0)$ $(i = 1, 2)$ in the Navier type boundary conditions on Γ_0:

$$\left\{ \begin{array}{rcll} \boldsymbol{z}_i \cdot \mathbf{n} & = & 0 & \text{on } \Gamma_0, \\ 2\nu \left[\mathcal{D}(\boldsymbol{z}_i)\mathbf{n}\right]_\tau + \alpha_i \boldsymbol{z}_i & = & \mathbf{0} & \text{on } \Gamma_0. \end{array} \right. \tag{1.13}$$

We also consider the reconstruction of the Robin coefficient, still denoted α, in the case of the classical Robin boundary conditions on Γ_0 given by:

$$\sigma(\boldsymbol{z}_i, \pi_i)\mathbf{n} + \alpha_i \boldsymbol{z}_i = \mathbf{0} \quad \text{on } \Gamma_0. \tag{1.14}$$

For these two boundary conditions, we obtain (see Section 4) the following

Theorem 1.3. *Let* $(\boldsymbol{z}_i, \pi_i) \in \mathbf{H}^2(\Omega) \times \mathrm{H}^1(\Omega)$, $i = 1, 2$, *be two pairs solution of the Navier-Stokes equations* (1.12) *with the boundary conditions* (1.13) *or* (1.14) *which satisfy* (1.3) *for some* $K \geq e^e$. *Let* $\mathcal{N} \overset{def}{=} \{x \in \Gamma_0\,,\ \boldsymbol{z}_1(x) = \mathbf{0}$ *and* $\boldsymbol{z}_2(x) = \mathbf{0}\}$, $\mathcal{K}$ *be a compact subset of* $\Gamma_0 \backslash \mathcal{N}$ *and let* $m > 0$ *be a constant such that* $\max(|\boldsymbol{z}_1|, |\boldsymbol{z}_2|) \geq m$ *on* $\mathcal{K}$. *Then the following inequality holds:*

$$\begin{aligned} &\|\alpha_1 - \alpha_2\|_{\mathrm{L}^2(\mathcal{K})} \\ &\leq \frac{C}{m} \frac{K \left(\|\boldsymbol{z}_1 - \boldsymbol{z}_2\|_{\mathbf{H}^2(\Omega)} + \|\pi_1 - \pi_2\|_{\mathrm{H}^1(\Omega)}\right)}{\left(\ln\left(1 + \dfrac{\|\boldsymbol{z}_1 - \boldsymbol{z}_2\|_{\mathbf{H}^2(\Omega)} + \|\pi_1 - \pi_2\|_{\mathrm{H}^1(\Omega)}}{\|\boldsymbol{z}_1 - \boldsymbol{z}_2\|_{\mathbf{H}^{3/2}(\Gamma_{\text{obs}})} + \|\sigma(\boldsymbol{z}_1, \pi_1)\mathbf{n} - \sigma(\boldsymbol{z}_2, \pi_2)\mathbf{n}\|_{\mathbf{H}^{1/2}(\Gamma_{\text{obs}})}}\right)\right)^{1/4}}. \end{aligned} \tag{1.15}$$

Remark 1.4. *Since the assumptions of Theorem 1.3 guarantee that* $\boldsymbol{z}_1$, $\boldsymbol{z}_1$ *are continuous, the constant* $m > 0$ *always exists and depends on* $\boldsymbol{z}_1$, $\boldsymbol{z}_1$ *on* $\mathcal{K}$. *Obviously, the relevant case is* $\mathcal{K}$ *with nonempty interior, i.e.* $\mathcal{N} \neq \Gamma_0$. *In fact, in the case of Robin boundary conditions* (1.14) *and if* $\boldsymbol{z}_1$ *(or* $\boldsymbol{z}_2$*) is not identically equal to zero in* Ω, *we know that the interior of* $\mathcal{N}$ *is empty. It is an easy consequence of Fabre and Lebeau's theorem. But in the case of Navier conditions* (1.13) *and if one of the* $\boldsymbol{z}_i$ *is not trivial, the existence of a nonempty open subset of* Γ_0 *on which* $\boldsymbol{z}_1$ *and* $\boldsymbol{z}_2$ *both vanish is a difficult issue. Indeed,* *it reduces to study the existence of a non trivial vector field* $\boldsymbol{v}$ *solution to an homogeneous Oseen equation (see* (4.1) *below) and such that* $\boldsymbol{v} = \partial_{\mathbf{n}}\boldsymbol{v} = \mathbf{0}$ *on a nonempty open subset of* Γ_0*. The difficulty relies on the fact that, unlike the Robin case, no additional information on the pressure is available.*

Remark 1.5. *We can obtain a better estimate assuming more regularity on* $(\boldsymbol{v}, p)$*. More precisely, for* $k \geq 2$ *and* $n \in \mathbb{N}$ *suppose that* $(\boldsymbol{v}, p) \in \mathbf{H}^k(\Omega) \times \mathrm{H}^{k-1}(\Omega)$*,* $k \geq 2$ *and* $\alpha_i \in \mathrm{H}^n(\mathcal{K})$*,* $i = 1, 2$*. Then, using an interpolation argument, we can obtain for all* $\theta \in [0, 1]$ *(see Remark 4.1):*

$$\|\alpha_1 - \alpha_2\|_{\mathrm{H}^{\theta n}(\mathcal{K})}$$
$$\leq \frac{\left(\frac{CK}{m}\|\boldsymbol{v}\|_{\mathbf{H}^k(\Omega)} + \|p\|_{\mathrm{H}^{k-1}(\Omega)}\right)^{1-\theta}\|\alpha_1 - \alpha_2\|^{\theta}_{\mathrm{H}^n(\mathcal{K})}}{\left(\ln\left(1 + \frac{\|\boldsymbol{v}\|_{\mathbf{H}^2(\Omega)} + \|p\|_{\mathrm{H}^1(\Omega)}}{\|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_{\mathbf{H}^{3/2}(\Gamma_{\mathrm{obs}})} + \|\sigma(\boldsymbol{v}_1, p_1)\mathbf{n} - \sigma(\boldsymbol{v}_2, p_2)\mathbf{n}\|_{\mathbf{H}^{1/2}(\Gamma_{\mathrm{obs}})}}\right)\right)^{\frac{(2k-3)(1-\theta)}{2k}}}. \tag{1.16}$$

For $k = 3$ *and* $\theta = n = 0$*, we obtain a result similar to the one presented in [8, Theorem 4.3].*

Theorem 1.3, which completes the previous results given by Boulakia *et al* in [8, 9], finds applications in the modeling of biological problems as blood flow in the cardiovascular system (see [26] and [29]) or airflow in the lungs (see [4]). For the Laplace equation, these kind of stability estimates for the Robin coefficient have been widely studied: see for example the works of Chaabane *et al.* in [13, 12], Alessandrini *et al.* in [2], Sincich in [27], Bellassoued *et al.* in [6] and Cheng *et al.* in [14].

Finally, we present another application of our stability estimates in the context of numerical reconstruction methods. More precisely, we focus on the stable reconstruction of the solution of a data completion problem (also known as Cauchy problem) for the Stokes equations: for given $(\boldsymbol{g}_D, \boldsymbol{g}_N) \in \mathbf{H}^{3/2}(\Gamma_{\mathrm{obs}}) \times \mathbf{H}^{1/2}(\Gamma_{\mathrm{obs}})$, we search $(\boldsymbol{v}, p) \in \mathbf{H}^2(\Omega) \times \mathrm{H}^1(\Omega)$ solution of

$$\left\{\begin{array}{rcll} -\nu\Delta\boldsymbol{v} + \nabla p &=& \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{v} &=& 0 & \text{in } \Omega, \end{array}\right. \tag{1.17}$$

and such that

$$\boldsymbol{v} = \boldsymbol{g}_D \quad \text{and} \quad \sigma(\boldsymbol{v}, p)\mathbf{n} = \boldsymbol{g}_N \quad \text{on} \quad \Gamma_{\mathrm{obs}}.$$

Estimates (1.5) and (1.6) imply the uniqueness of the solution of the data completion problem. However, there exists Cauchy data $(\boldsymbol{g}_D, \boldsymbol{g}_N)$ for which it does not admit any solution. Hence, regularization methods are needed to stably reconstruct $(\boldsymbol{v}, p)$ from $(\boldsymbol{g}_D, \boldsymbol{g}_N)$. We study two standard regularization methods: a quasi-reversibility regularization and a penalized Kohn-Vogelius regularization.

In the quasi-reversibility method, we consider, for $\varepsilon > 0$, the following variational problem: find $(\boldsymbol{v}_\varepsilon, p_\varepsilon) \in \mathbf{H}^2(\Omega) \times \mathrm{H}^1(\Omega)$ such that $\boldsymbol{v}_\varepsilon = \boldsymbol{g}_D$ on Γ_{obs}, $\sigma(\boldsymbol{v}_\varepsilon, p_\varepsilon)\mathbf{n} = \boldsymbol{g}_N$ on Γ_{obs}, and for all $(\boldsymbol{w}, q) \in \mathbf{H}^2(\Omega) \times \mathrm{H}^1(\Omega)$ such that $\boldsymbol{w} = \mathbf{0}$ and $\sigma(\boldsymbol{w}, q)\mathbf{n} = \mathbf{0}$ on Γ_{obs}, we have

$$\int_\Omega (-\nu\Delta\boldsymbol{v}_\varepsilon + \nabla p_\varepsilon) \cdot (-\nu\Delta\boldsymbol{w} + \nabla q)\,\mathrm{d}x + \Big(\operatorname{div}(\boldsymbol{v}_\varepsilon), \operatorname{div}(\boldsymbol{w})\Big)_{\mathrm{H}^1(\Omega)}$$
$$+ \varepsilon(\boldsymbol{v}_\varepsilon, \boldsymbol{w})_{\mathbf{H}^2(\Omega)} + \varepsilon(p_\varepsilon, q)_{\mathrm{H}^1(\Omega)} = \int_\Omega \boldsymbol{f} \cdot (-\nu\Delta\boldsymbol{w} + \nabla q)\,\mathrm{d}x. \tag{1.18}$$

The penalized Kohn-Vogelius approach that we consider here consists in, for $\varepsilon > 0$ and $\Gamma_{\text{obs}}^C \overset{\text{def}}{=} \partial\Omega \backslash \overline{\Gamma_{\text{obs}}}$, defining the functional $F_\varepsilon : \mathbf{H}^{1/2}(\Gamma_{\text{obs}}^C) \times \mathbf{H}^{3/2}(\Gamma_{\text{obs}}^C) \to \mathbb{R}$ given by

$$\begin{aligned} F_\varepsilon(\boldsymbol{\varphi}_N, \boldsymbol{\psi}_D) \overset{\text{def}}{=} & |\boldsymbol{v}_{\boldsymbol{\varphi}_N} - \boldsymbol{v}_{\boldsymbol{\psi}_D}|^2_{\mathbf{H}^2(\Omega)} + |\boldsymbol{v}_{\boldsymbol{\varphi}_N} - \boldsymbol{v}_{\boldsymbol{\psi}_D}|^2_{\mathbf{H}^1(\Omega)} \\ & + \varepsilon \|\boldsymbol{v}_{\boldsymbol{\varphi}_N}, p_{\boldsymbol{\varphi}_N}\|^2_{\mathbf{H}^2(\Omega)\times \mathrm{H}^1(\Omega)} + \varepsilon \|\boldsymbol{v}_{\boldsymbol{\psi}_D}, p_{\boldsymbol{\psi}_D}\|^2_{\mathbf{H}^2(\Omega)\times \mathrm{H}^1(\Omega)}, \end{aligned}$$

where $|.|_{\mathbf{H}^i(\Omega)}$ is the $\mathbf{H}^i$-seminorm ($i = 1, 2$) and where $(\boldsymbol{v}_{\boldsymbol{\varphi}_N}, p_{\boldsymbol{\varphi}_N}) \in \mathbf{H}^2(\Omega) \times \mathrm{H}^1(\Omega)$ and $(\boldsymbol{v}_{\boldsymbol{\psi}_D}, p_{\boldsymbol{\psi}_D}) \in \mathbf{H}^2(\Omega) \times \mathrm{H}^1(\Omega)$ are the respective solutions of

$$\left\{ \begin{array}{rcll} -\nu\Delta \boldsymbol{v}_{\boldsymbol{\varphi}_N} + \nabla p_{\boldsymbol{\varphi}_N} & = & \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{v}_{\boldsymbol{\varphi}_N} & = & 0 & \text{in } \Omega, \\ \boldsymbol{v}_{\boldsymbol{\varphi}_N} & = & \boldsymbol{g}_D & \text{on } \Gamma_{\text{obs}}, \\ \sigma(\boldsymbol{v}_{\boldsymbol{\varphi}_N}, p_{\varphi})\mathbf{n} & = & \boldsymbol{\varphi}_N & \text{on } \Gamma_{\text{obs}}^C, \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{rcll} -\nu\Delta \boldsymbol{v}_{\boldsymbol{\psi}_D} + \nabla p_{\boldsymbol{\psi}_D} & = & \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{v}_{\boldsymbol{\psi}_D} & = & 0 & \text{in } \Omega, \\ \sigma(\boldsymbol{v}_{\boldsymbol{\psi}_D}, p_{\boldsymbol{\psi}_D})\mathbf{n} & = & \boldsymbol{g}_N & \text{on } \Gamma_{\text{obs}}, \\ \boldsymbol{v}_{\boldsymbol{\psi}_D} & = & \boldsymbol{\psi}_D & \text{on } \Gamma_{\text{obs}}^C. \end{array} \right.$$

Then, we define $(\boldsymbol{v}_\varepsilon, p_\varepsilon) \overset{\text{def}}{=} (\boldsymbol{v}_{\boldsymbol{\varphi}_N^\varepsilon}, p_{\boldsymbol{\psi}_D^\varepsilon})$ where $(\boldsymbol{\varphi}_N^\varepsilon, \boldsymbol{\psi}_D^\varepsilon) \in \mathbf{H}^{1/2}(\Gamma_{\text{obs}}^C) \times \mathbf{H}^{3/2}(\Gamma_{\text{obs}}^C)$ is such that

$$F_\varepsilon(\boldsymbol{\varphi}_N^\varepsilon, \boldsymbol{\psi}_D^\varepsilon) = \inf_{(\boldsymbol{\varphi}_N, \boldsymbol{\psi}_D) \in \mathbf{H}^{1/2}(\Gamma_{\text{obs}}^C) \times \mathbf{H}^{3/2}(\Gamma_{\text{obs}}^C)} F_\varepsilon(\boldsymbol{\varphi}_N, \boldsymbol{\psi}_D). \tag{1.19}$$

For any $(\boldsymbol{g}_D, \boldsymbol{g}_N) \in \mathbf{H}^{3/2}(\Gamma_{\text{obs}}) \times \mathbf{H}^{1/2}(\Gamma_{\text{obs}})$, both the quasi-reversibility problem (1.18) and the Kohn-Vogelius minimization problem (1.19) admits a unique solution $(\boldsymbol{v}_\varepsilon, p_\varepsilon)$. Moreover, if the initial data completion problem admits a solution $(\boldsymbol{v}, p)$, then $\boldsymbol{v}_\varepsilon$ converges to $\boldsymbol{v}$ strongly in $\mathbf{H}^2(\Omega)$ and p_ε converges to p strongly in $\mathrm{H}^1(\Omega)$. Furthermore, the stability estimates we obtain in the present paper (proved in Section 5) provide the rate of convergence of both methods (for a survey on the connection between stability estimates and rates of convergence of regularization methods, we refer to [21]):

Theorem 1.6. *There exist $C(\boldsymbol{v}, p)$ and $\widetilde{C}(\boldsymbol{v}, p)$, two constants depending only on Ω and on the $\mathbf{H}^2 \times \mathbf{H}^1$ norm of the exact solution of the data completion problem* (1.17)*, such that we have the following error estimates for both quasi-reversibility method and penalized Kohn-Vogelius method:*

$$\|\boldsymbol{v}_\varepsilon - \boldsymbol{v}\|_{\mathbf{L}^2(\Omega)} \leq \frac{C(\boldsymbol{v}, p)}{\ln(1 + \frac{\widetilde{C}(\boldsymbol{v},p)}{\sqrt{\varepsilon}})}, \quad \|\boldsymbol{v}_\varepsilon - \boldsymbol{v}\|_{\mathbf{H}^1(\Omega)} \leq \frac{C(\boldsymbol{v}, p)}{\left(\ln(1 + \frac{\widetilde{C}(\boldsymbol{v},p)}{\sqrt{\varepsilon}})\right)^{1/2}}$$

and

$$\|p_\varepsilon - p\|_{\mathrm{L}^2(\Omega)} \leq \frac{C(\boldsymbol{u}, p)}{\left(\ln(1 + \frac{\widetilde{C}(\boldsymbol{v},p)}{\sqrt{\varepsilon}})\right)^{1/2}}.$$

Notations. All along this paper, Ω is a nonempty bounded open subset of $\mathbb{R}^N$ ($N = 2$ or $N = 3$) with a boundary $\partial\Omega$ of class C^2, ω is a nonempty open subset of Ω, Γ_{obs} and Γ_0 are nonempty open subsets of $\partial\Omega$, $\Gamma_{\text{obs}} \cap \Gamma_0 = \emptyset$, and Γ_{obs}^C denotes the complement of Γ_{obs}, namely $\Gamma_{\text{obs}}^C \overset{\text{def}}{=} \partial\Omega \backslash \overline{\Gamma_{\text{obs}}}$.

We here summarize the needed notations in the case $N = 3$ which can be easily adapted for $N = 2$. We denote by $\mathbf{n} = {}^t(\mathrm{n}_1, \mathrm{n}_2, \mathrm{n}_3)$ the outward unit normal to $\partial\Omega$ (where ^{t} denotes the transpose). For a scalar function w or a vector field $\boldsymbol{y} = {}^t(y_1, y_2, y_3)$, we define $\nabla w \overset{\text{def}}{=} {}^t(\partial_{x_1} w, \partial_{x_2} w, \partial_{x_3} w)$, $\nabla \boldsymbol{y} \overset{\text{def}}{=} (\partial_{x_j} y_i)_{1 \leq i,j \leq 3}$ and $\operatorname{div} \boldsymbol{y} \overset{\text{def}}{=} \sum_{i=1}^3 \partial_{x_i} y_i$. Moreover, on $\partial\Omega$, we define the normal derivatives $\frac{\partial w}{\partial \mathbf{n}} \overset{\text{def}}{=} (\nabla w) \cdot \mathbf{n}$ and $\frac{\partial \boldsymbol{y}}{\partial \mathbf{n}} \overset{\text{def}}{=} (\nabla \boldsymbol{y})\mathbf{n}$ and the tangential gradients $\nabla_\tau w \overset{\text{def}}{=} \nabla w - \frac{\partial w}{\partial \mathbf{n}}\mathbf{n}$ and $\nabla_\tau \boldsymbol{y} \overset{\text{def}}{=} {}^t(\nabla_\tau y_1, \nabla_\tau y_2, \nabla_\tau y_3)$. We also introduce the notations $\boldsymbol{y}_{\rm n} \overset{\text{def}}{=} (\boldsymbol{y}\cdot\mathbf{n})\,\mathbf{n}$ and $\boldsymbol{y}_\tau \overset{\text{def}}{=} \boldsymbol{y} - \boldsymbol{y}_{\rm n}$ for the normal and the tangential components of $\boldsymbol{y}$ on $\partial\Omega$. The divergence of $\boldsymbol{y}$ is defined by $\operatorname{div}\boldsymbol{y} \overset{\text{def}}{=} \sum_{j=1}^3 \partial_{x_j} y_j$ and the curl of w or $\boldsymbol{y}$ is defined by

$$\operatorname{curl}\boldsymbol{y} = \partial_{x_1} y_2 - \partial_{x_2} y_1 \quad \text{and} \quad \operatorname{curl} w = \begin{pmatrix} \partial_{x_2} w \\ -\partial_{x_1} w \end{pmatrix} \quad \text{if } N = 2,$$

and

$$\operatorname{curl}\boldsymbol{y} \overset{\text{def}}{=} \begin{pmatrix} \partial_{x_2} y_3 - \partial_{x_3} y_2 \\ \partial_{x_3} y_1 - \partial_{x_1} y_3 \\ \partial_{x_1} y_2 - \partial_{x_2} y_1 \end{pmatrix} \quad \text{if } N = 3.$$

We will also need to use the tangential divergence operator on $\partial\Omega$ that we denote by div_τ. We recall that $\mathcal{D}(\boldsymbol{y}) \overset{\text{def}}{=} \frac{1}{2}\left(\nabla\boldsymbol{y} + {}^t\nabla\boldsymbol{y}\right)$ denotes the symmetrized gradient and $\sigma(\boldsymbol{y}, p) \overset{\text{def}}{=} 2\nu\mathcal{D}(\boldsymbol{y}) - p\,\mathrm{I}$ the stress tensor, where I denotes the identity matrix and $\nu > 0$ is the constant which represents the kinematic viscosity of the fluid we consider.

For $r \geq 0$ we denote by $\mathrm{L}^2(\Omega)$, $\mathrm{L}^2(\partial\Omega)$, $\mathrm{H}^r(\Omega)$, $\mathrm{H}^r(\partial\Omega)$, $\mathrm{H}^r_0(\Omega)$, the usual Lebesgue and Sobolev spaces of scalar functions in Ω or in $\partial\Omega$, and we write in bold the spaces of vector-valued functions: $\mathbf{L}^2(\Omega) = (\mathrm{L}^2(\Omega))^N$, $\mathbf{L}^2(\partial\Omega) = (\mathrm{L}^2(\partial\Omega))^N$, etc.

We recall that $\boldsymbol{z}_1$, $\boldsymbol{z}_2$ are vector fields satisfying (1.2). Moreover, we use the following particular constant:

$$\mathfrak{m}(\boldsymbol{z}_1, \boldsymbol{z}_2) \overset{\text{def}}{=} \max\left\{1\,,\, \|\boldsymbol{z}_1\|_{\mathbf{L}^\infty(\Omega)}\,,\, \|\nabla\boldsymbol{z}_2\|_{\mathrm{L}^r(\Omega)}\right\}. \tag{1.20}$$

We also recall that $C > 0$ denotes a generic constant only depending on the geometry. In particular, it is independent on $\boldsymbol{z}_1$, $\boldsymbol{z}_2$ and on the parameters s, λ appearing in Carleman inequalities of sections 2 and 3.

Finally, for $\mathcal{O}_1$, $\mathcal{O}_2$ two open subsets of $\mathbb{R}^N$, the notation $\mathcal{O}_1 \Subset \mathcal{O}_2$ means that there exists a compact set $\mathcal{K}$ such that $\mathcal{O}_1 \subset \mathcal{K} \subset \mathcal{O}_2$.

Organization of the paper. The paper is organized as follows. The Section 2 is dedicated to the proof of Carleman inequalities for the non-homogeneous Oseen equations (see Theorem 2.3). It is obtained by combining a domain extension argument with Carleman inequalities for compactly supported solutions of the Stokes equations. Then in Section 3, we deduce a Hölder type interior estimates for a distributed observation as well as log type stability inequalities for both distributed and boundary observations. In particular, Theorem 1.1 is proved in subsections 3.1 and 3.2 and Theorem 1.2 is proved in subsection 3.3. Finally, we present some applications in the last sections. The Section 4 concerns the proof of stability inequalities for the inverse problem of recovering Navier and Robin coefficients (proof of Theorem 1.3) and Section 5 is dedicated to the proof of error estimates for some numerical reconstruction methods (proof of Theorem 1.6).

2 Carleman Inequality for Stokes and Oseen equations

In this section, $\mathcal{O}$ is a non empty bounded open subset of $\mathbb{R}^N$ ($N = 2$ or $N = 3$) of class C^2, ω is a non empty bounded open subset such that $\omega \Subset \mathcal{O}$ and $\psi : \mathcal{O} \to \mathbb{R}$ is a function satisfying

$$\begin{gathered} \psi \in C^2(\mathcal{O};\mathbb{R}), \quad \psi > 0 \quad \text{and} \quad |\nabla\psi| > 0 \quad \text{in } \mathcal{O}\backslash\overline{\omega} \\ \psi = c_0 \quad \text{on } \partial\mathcal{O}, \end{gathered} \tag{2.1}$$

for some positive constant $c_0 > 0$. For the existence of such a function see for instance [18] or [28, Appendix III]. Here, the set $\mathcal{O}$ plays the role of Ω or of an extension $\widetilde{\Omega}$ of Ω which is used in Section 3 below.

The main aim of this section is to prove a Carleman inequality for the non homogeneous Oseen equations. For that, we first prove a Carleman inequality for a pair velocity-pressure in $\mathbf{H}^2_0(\mathcal{O}) \times \mathrm{H}^1_0(\mathcal{O})$ and then we use a domain extension argument to recover the non-homogeneous case.

2.1 Carleman Inequality in the case of homogeneous boundary data

Let us first recall a standard Carleman inequality for the Laplace equation:

Theorem 2.1. *Let $k \in \{0, 1\}$, $F \in \mathrm{L}^2(\mathcal{O})$ and $\boldsymbol{G} \in \mathbf{L}^2(\mathcal{O})$. There exist $C > 0$, $\widehat{\lambda} > 1$ and $\widehat{s} > 1$ such that for all $\lambda \geq \widehat{\lambda}$ and $s \geq \widehat{s}$, the solution $u \in \mathrm{H}^1(\mathcal{O})$ of*

$$\left\{\begin{array}{rcll} -\Delta u &=& F + \operatorname{div} \boldsymbol{G} & \text{in } \mathcal{O}, \\ u &=& 0 & \text{on } \partial\mathcal{O}, \end{array}\right.$$

satisfies the following inequality:

$$\begin{aligned} &\int_{\mathcal{O}} \left(e^{(k-1)\lambda\psi} |\nabla u|^2 + s^2\lambda^2 e^{(k+1)\lambda\psi} |u|^2 \right) e^{2se^{\lambda\psi}} \mathrm{d}x \\ &\qquad \leq C \left(\int_{\mathcal{O}} \left(s e^{k\lambda\psi} |\boldsymbol{G}|^2 + s^{-1}\lambda^{-2} e^{(k-2)\lambda\psi} |F|^2 \right) e^{2se^{\lambda\psi}} \mathrm{d}x \right. \\ &\qquad\qquad \left. + \int_{\omega} s^2\lambda^2 e^{(k+1)\lambda\psi} |u|^2 e^{2se^{\lambda\psi}} \mathrm{d}x \right). \end{aligned} \tag{2.2}$$

Proof. Inequality (2.2) for $k = 1$ is given for instance in [20, Theorem A.1] and (2.2) for $k = 0$ is obtained by applying (2.2) with $k = 1$ to the equation satisfied by $e^{-\frac{\lambda}{2}\psi} u$. Note that the above quoted result is stated for a function ψ that vanishes on $\partial\mathcal{O}$. However, if $\widetilde{s}, \widetilde{\lambda}$ denote the admissible parameters of [20, Theorem A.1], it suffices to choose $(s, \lambda) = (\widetilde{s} e^{\widetilde{\lambda} c_0}, \widetilde{\lambda})$ to get (2.2). ☐

We deduce the following Carleman inequality for Stokes equations:

Theorem 2.2. *There exist $C > 0$, $\widehat{\lambda} > 1$ and $\widehat{s} > 1$ such that for all $\lambda \geq \widehat{\lambda}$ and $s \geq \widehat{s}$, and for all $(\boldsymbol{v}, p) \in \mathbf{H}^2_0(\mathcal{O}) \times \mathrm{H}^1_0(\mathcal{O})$ the following inequalities hold:*

$$\begin{aligned} &\int_{\mathcal{O}} s e^{\lambda\psi} |\operatorname{div} \boldsymbol{v} - p|^2 e^{2se^{\lambda\psi}} \mathrm{d}x \\ &\qquad \leq C \left(\int_{\omega} s e^{\lambda\psi} |\operatorname{div} \boldsymbol{v} - p|^2 e^{2se^{\lambda\psi}} \mathrm{d}x + \int_{\mathcal{O}} \lambda^{-2} |\nabla p - \Delta \boldsymbol{v}|^2 e^{2se^{\lambda\psi}} \mathrm{d}x \right) \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} &\int_{\mathcal{O}} \left(|\nabla \boldsymbol{v}|^2 + s e^{\lambda\psi} |\operatorname{curl} \boldsymbol{v}|^2 + s^2\lambda^2 e^{2\lambda\psi} |\boldsymbol{v}|^2 \right) e^{2se^{\lambda\psi}} \mathrm{d}x \\ &\leq C \left(\int_{\mathcal{O}} (s^{-1}\lambda^{-2} e^{-\lambda\psi} |\nabla \operatorname{div} \boldsymbol{v}|^2 + \lambda^{-2} |\nabla p - \Delta \boldsymbol{v}|^2) e^{2se^{\lambda\psi}} \mathrm{d}x + \int_{\omega} s^3\lambda^2 e^{3\lambda\psi} |\boldsymbol{v}|^2 e^{2se^{\lambda\psi}} \mathrm{d}x \right). \end{aligned} \tag{2.4}$$

Proof. First, we set $\boldsymbol{f} \overset{\text{def}}{=} -\Delta \boldsymbol{v} + \nabla p$. Easy calculations yield:

$$\begin{aligned} -\Delta(\operatorname{curl} \boldsymbol{v}) &= \operatorname{curl} \boldsymbol{f} \quad \text{in } \mathcal{O}, && (2.5)\\ -\Delta(\operatorname{div} \boldsymbol{v} - p) &= \operatorname{div} \boldsymbol{f} \quad \text{in } \mathcal{O}, && (2.6)\\ -\Delta \boldsymbol{v} &= \operatorname{curl}(\operatorname{curl} \boldsymbol{v}) - \nabla(\operatorname{div} \boldsymbol{v}) \quad \text{in } \mathcal{O}. && (2.7)\end{aligned}$$

Then, by applying (2.2) for $k = 0$ to (2.6) we obtain (2.3).

Next, we introduce another open subset $\omega_0 \Subset \omega$ and apply (2.2) for $k = 0$ to (2.5) to obtain:

$$\begin{aligned}\int_{\mathcal{O}} s^{-1}\lambda^{-2}e^{-\lambda\psi}|\nabla(\operatorname{curl} \boldsymbol{v})|^2 e^{2se^{\lambda\psi}}\mathrm{d}x + \int_{\mathcal{O}} se^{\lambda\psi}|\operatorname{curl} \boldsymbol{v}|^2 e^{2se^{\lambda\psi}}\mathrm{d}x \\ \leq C\left(\int_{\omega_0} se^{\lambda\psi}|\operatorname{curl} \boldsymbol{v}|^2 e^{2se^{\lambda\psi}}\mathrm{d}x + \int_{\mathcal{O}} \lambda^{-2}|\nabla p - \Delta \boldsymbol{v}|^2 e^{2se^{\lambda\psi}}\mathrm{d}x\right). \quad (2.8)\end{aligned}$$

Let us replace the local term in $\operatorname{curl} \boldsymbol{v}$ by a local term in $\boldsymbol{v}$. For that, we introduce a function $\rho \in C_c^\infty(\omega)$ such that $0 \leq \rho \leq 1$ and $\rho = 1$ in ω_0. Using an integration by parts in ω, we get

$$\begin{aligned}\int_{\omega_0} se^{\lambda\psi}|\operatorname{curl} \boldsymbol{v}|^2 e^{2se^{\lambda\psi}}\mathrm{d}x \leq s\int_{\omega} \rho e^{\lambda\psi}|\operatorname{curl} \boldsymbol{v}|^2 e^{2se^{\lambda\psi}}\mathrm{d}x = s\int_{\omega} \operatorname{curl}\left(\rho e^{\lambda\psi}e^{2se^{\lambda\psi}}\operatorname{curl} \boldsymbol{v}\right)\boldsymbol{v}\,\mathrm{d}x \\ \leq C\left(\int_{\omega} s^2\lambda e^{2\lambda\psi}e^{2se^{\lambda\psi}}\,|\boldsymbol{v}|\,|\operatorname{curl} \boldsymbol{v}|\,\mathrm{d}x + \int_{\omega} se^{\lambda\psi}e^{2se^{\lambda\psi}}\,|\nabla(\operatorname{curl} \boldsymbol{v})|\,|\boldsymbol{v}|\,\mathrm{d}x\right),\end{aligned}$$

and with Cauchy-Schwarz inequality:

$$\begin{aligned}\int_{\omega_0} se^{\lambda\psi}|\operatorname{curl} \boldsymbol{v}|^2 e^{2se^{\lambda\psi}}\mathrm{d}x \leq \epsilon\int_{\mathcal{O}}\left(s^{-1}\lambda^{-2}e^{-\lambda\psi}e^{2se^{\lambda\psi}}\,|\nabla(\operatorname{curl} \boldsymbol{v})|^2 + se^{\lambda\psi}e^{2se^{\lambda\psi}}\,|\operatorname{curl} \boldsymbol{v}|^2\right)\mathrm{d}x \\ + \frac{C}{\epsilon}s^3\lambda^2\int_{\omega} e^{3\lambda\psi}e^{2se^{\lambda\psi}}\,|\boldsymbol{v}|^2\,\mathrm{d}x.\end{aligned}$$

By combining (2.8) with the above inequality for $\epsilon > 0$ small enough, we obtain

$$\begin{aligned}\int_{\mathcal{O}} s^{-1}\lambda^{-2}e^{-\lambda\psi}|\nabla(\operatorname{curl} \boldsymbol{v})|^2 e^{2se^{\lambda\psi}}\mathrm{d}x + \int_{\mathcal{O}} se^{\lambda\psi}|\operatorname{curl} \boldsymbol{v}|^2 e^{2se^{\lambda\psi}}\mathrm{d}x \\ \leq C\left(\int_{\omega} s^3\lambda^2 e^{3\lambda\psi}|\boldsymbol{v}|^2 e^{2se^{\lambda\psi}}\mathrm{d}x + \int_{\mathcal{O}} \lambda^{-2}|\nabla p - \Delta \boldsymbol{v}|^2 e^{2se^{\lambda\psi}}\mathrm{d}x\right). \quad (2.9)\end{aligned}$$

Finally, (2.4) is obtained by first applying (2.2) for $k = 1$ to (2.7) and next using the estimate of $\operatorname{curl} \boldsymbol{v}$ given by (2.9). □

2.2 Carleman Inequality in the case of non-homogeneous boundary data

In this section, we prove a Carleman inequality for the Oseen equations:

$$\left\{\begin{aligned} -\nu\Delta \boldsymbol{v} + (\boldsymbol{z}_1 \cdot \nabla)\boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{z}_2 + \nabla p &= \boldsymbol{f} \quad \text{in } \mathcal{O}, \\ \operatorname{div} \boldsymbol{v} &= d \quad \text{in } \mathcal{O}. \end{aligned}\right. \qquad (2.10)$$

Above and in the following, $\boldsymbol{z}_1 \in \mathbf{L}^\infty(\mathcal{O})$, $\boldsymbol{z}_2 \in \mathbf{W}^{1,r}(\mathcal{O})$ (with $r > 2$ if $N = 2$ and $r = 3$ if $N = 3$) and we use the following notation for the particular constant:

$$\widetilde{\mathfrak{m}}(\boldsymbol{z}_1, \boldsymbol{z}_2) \overset{\text{def}}{=} \max\left\{1\,,\, \|\boldsymbol{z}_1\|_{\mathbf{L}^\infty(\mathcal{O})}\,,\, \|\nabla \boldsymbol{z}_2\|_{\mathrm{L}^r(\mathcal{O})}\right\}. \qquad (2.11)$$

We recall that $C > 0$ denote a generic constant only depending on the geometry and independent on s, λ, $\boldsymbol{z}_1$, $\boldsymbol{z}_2$.

Theorem 2.3. *There exist $C > 0$, $\widehat{c} > 0$ and $\widehat{s} > 1$ such that for all $\boldsymbol{z}_1 \in \mathbf{L}^\infty(\mathcal{O})$, $\boldsymbol{z}_2 \in \mathbf{W}^{1,r}(\mathcal{O})$, all $\lambda \geq \widehat{\lambda} \overset{def}{=} \widetilde{\mathfrak{m}}(\boldsymbol{z}_1, \boldsymbol{z}_2)\widehat{c}$ and all $s \geq \widehat{s}$ every solution $(\boldsymbol{v}, p) \in \mathbf{H}^2(\mathcal{O}) \times \mathrm{H}^1(\mathcal{O})$ of* (2.10) *satisfies:*

$$\begin{aligned}\int_{\mathcal{O}} &\left(|\nabla \boldsymbol{v}|^2 + se^{\lambda\psi}|\operatorname{curl} \boldsymbol{v}|^2 + s^2\lambda^2 e^{2\lambda\psi}|\boldsymbol{v}|^2\right) e^{2se^{\lambda\psi}} \mathrm{d}x \\ &\leq C \left(\int_{\mathcal{O}} (s^{-1}\lambda^{-2}e^{-\lambda\psi}|\nabla d|^2 + \lambda^{-2}|\boldsymbol{f}|^2) e^{2se^{\lambda\psi}} \mathrm{d}x + \int_\omega s^3\lambda^2 e^{3\lambda\psi}|\boldsymbol{v}|^2 e^{2se^{\lambda\psi}} \mathrm{d}x \right. \\ &\qquad \left. + e^{2se^{\lambda c_0}} \left(\|\boldsymbol{v}\|^2_{\mathbf{H}^2(\mathcal{O})} + \|p\|^2_{\mathrm{H}^1(\mathcal{O})} \right) \right) \end{aligned} \tag{2.12}$$

and

$$\begin{aligned}\int_{\mathcal{O}} se^{\lambda\psi}|p - d|^2 e^{2se^{\lambda\psi}} \mathrm{d}x &\leq C \left(\int_\omega (s^3\lambda^2 e^{3\lambda\psi}|\boldsymbol{v}|^2 + se^{\lambda\psi}|p - d|^2) e^{2se^{\lambda\psi}} \mathrm{d}x \right. \\ &\left. + \int_{\mathcal{O}} (s^{-1}\lambda^{-2}e^{-\lambda\psi}|\nabla d|^2 + \lambda^{-2}|\boldsymbol{f}|^2) e^{2se^{\lambda\psi}} \mathrm{d}x + e^{2se^{\lambda c_0}} \left(\|\boldsymbol{v}\|^2_{\mathbf{H}^2(\mathcal{O})} + \|p\|^2_{\mathrm{H}^1(\mathcal{O})} \right) \right). \end{aligned} \tag{2.13}$$

Proof. Let $\widetilde{\mathcal{O}}$ be a bounded domain of $\mathbb{R}^N$ of class C^2 such that $\mathcal{O} \Subset \widetilde{\mathcal{O}}$. We extend ψ to $\widetilde{\mathcal{O}}$ (while keeping the same name) in a such a way that:

$$\begin{gathered} \psi \in C^2(\widetilde{\mathcal{O}}; \mathbb{R}), \quad \psi > 0 \quad \text{and} \quad |\nabla\psi| > 0 \quad \text{in } \widetilde{\mathcal{O}}\backslash\overline{\omega}, \\ \psi \equiv c_0 \quad \text{on } \partial\mathcal{O}, \quad 0 < \psi < c_0 \quad \text{in } \widetilde{\mathcal{O}}\backslash\overline{\mathcal{O}}, \quad c_0 < \psi \quad \text{in } \mathcal{O}. \end{gathered} \tag{2.14}$$

Let $E : \mathbf{H}^2(\mathcal{O}) \times \mathrm{H}^1(\mathcal{O}) \to \mathbf{H}^2_0(\widetilde{\mathcal{O}}) \times \mathrm{H}^1_0(\widetilde{\mathcal{O}})$ be a linear continuous map (given for example by Stein's theorem, see [1]), also continuous from $\mathbf{H}^1(\mathcal{O}) \times \mathrm{L}^2(\mathcal{O})$ into $\mathbf{H}^1_0(\widetilde{\mathcal{O}}) \times \mathrm{L}^2(\widetilde{\mathcal{O}})$, such that $E(\boldsymbol{v}, p) \equiv (\boldsymbol{v}, p)$ in $\mathcal{O}$, and define $(\widetilde{\boldsymbol{v}}, \widetilde{p}) \overset{\text{def}}{=} E(\boldsymbol{v}, p)$. We also denote by $\widetilde{\boldsymbol{z}}_1$, $\widetilde{\boldsymbol{z}}_2$ some continuous extensions of $\boldsymbol{z}_1$, $\boldsymbol{z}_2$ for the $\mathbf{L}^\infty$ and $\mathbf{W}^{1,r}$ norms in $\widetilde{\mathcal{O}}$ respectively. The pair $(\widetilde{\boldsymbol{v}}, \widetilde{p}) \overset{\text{def}}{=} E(\boldsymbol{v}, p)$ is then solution to

$$\left\{ \begin{aligned} -\nu\Delta\widetilde{\boldsymbol{v}} + (\widetilde{\boldsymbol{z}}_1 \cdot \nabla)\,\widetilde{\boldsymbol{v}} + (\widetilde{\boldsymbol{v}} \cdot \nabla)\,\widetilde{\boldsymbol{z}}_2 + \nabla\widetilde{p} &= \widetilde{\boldsymbol{f}} \quad \text{in } \widetilde{\mathcal{O}}, \\ \operatorname{div} \widetilde{\boldsymbol{v}} &= \widetilde{d} \quad \text{in } \widetilde{\mathcal{O}}, \\ \widetilde{\boldsymbol{v}} &= \mathbf{0} \quad \text{on } \partial\widetilde{\mathcal{O}}, \\ \frac{\partial\widetilde{\boldsymbol{v}}}{\partial\mathbf{n}} &= \mathbf{0} \quad \text{on } \partial\widetilde{\mathcal{O}}, \\ \widetilde{p} &= 0 \quad \text{on } \partial\widetilde{\mathcal{O}}, \end{aligned} \right. \tag{2.15}$$

where $\widetilde{\boldsymbol{f}} \in \mathbf{L}^2(\widetilde{\mathcal{O}})$ and $\widetilde{d} \in \mathrm{H}^1(\widetilde{\mathcal{O}})$ are given by $\widetilde{\boldsymbol{f}} = \boldsymbol{f}$ and $\widetilde{d} = d$ in $\mathcal{O}$ and by $\widetilde{\boldsymbol{f}} = -\nu\Delta\widetilde{\boldsymbol{v}} + (\widetilde{\boldsymbol{z}}_1 \cdot \nabla)\,\widetilde{\boldsymbol{v}} + (\widetilde{\boldsymbol{v}} \cdot \nabla)\,\widetilde{\boldsymbol{z}}_2 + \nabla\widetilde{p}$ and $\widetilde{d} = \operatorname{div} \widetilde{\boldsymbol{v}}$ in $\widetilde{\mathcal{O}}\backslash\overline{\mathcal{O}}$. From the continuity of the extension operator E we have:

$$\|\widetilde{\boldsymbol{f}}\|_{\mathbf{L}^2(\widetilde{\mathcal{O}})} + \|\widetilde{d}\|_{\mathrm{H}^1(\widetilde{\mathcal{O}})} \leq C\,\widetilde{\mathfrak{m}}(\boldsymbol{z}_1, \boldsymbol{z}_2) \left(\|\boldsymbol{v}\|_{\mathbf{H}^2(\mathcal{O})} + \|p\|_{\mathrm{H}^1(\mathcal{O})} \right). \tag{2.16}$$

Next, by applying estimate (2.4) of Theorem 2.2:

$$\begin{aligned}\int_{\widetilde{\mathcal{O}}} &\left(|\nabla \widetilde{\boldsymbol{v}}| + se^{\lambda\psi}|\operatorname{curl} \widetilde{\boldsymbol{v}}|^2 + s^2\lambda^2 e^{2\lambda\psi}|\widetilde{\boldsymbol{v}}|^2\right) e^{2se^{\lambda\psi}} \mathrm{d}x \\ &\leq C \left(\int_{\widetilde{\mathcal{O}}} (s^{-1}\lambda^{-2}e^{-\lambda\psi}|\nabla \widetilde{d}|^2 + \lambda^{-2}|\widetilde{\boldsymbol{f}} - (\widetilde{\boldsymbol{z}}_1 \cdot \nabla)\,\widetilde{\boldsymbol{v}} - (\widetilde{\boldsymbol{v}} \cdot \nabla)\,\widetilde{\boldsymbol{z}}_2|^2) e^{2se^{\lambda\psi}} \mathrm{d}x \right. \\ &\qquad \left. + \int_\omega s^3\lambda^2 e^{3\lambda\psi}|\widetilde{\boldsymbol{v}}|^2 e^{2se^{\lambda\psi}} \mathrm{d}x \right). \end{aligned} \tag{2.17}$$

Since $\widetilde{\boldsymbol{z}}_1 \in \mathbf{L}^\infty(\widetilde{\mathcal{O}})$, we get

$$\int_{\widetilde{\mathcal{O}}} \lambda^{-2} \left|(\widetilde{\boldsymbol{z}}_1 \cdot \nabla)\, \widetilde{\boldsymbol{v}}\right|^2 e^{2se^{\lambda\psi}} \leq \|\widetilde{\boldsymbol{z}}_1\|^2_{\mathbf{L}^\infty(\widetilde{\mathcal{O}})} \lambda^{-2} \int_{\widetilde{\mathcal{O}}} |\nabla \widetilde{\boldsymbol{v}}|^2 e^{2se^{\lambda\psi}}. \tag{2.18}$$

Moreover, since $\widetilde{\boldsymbol{z}}_2 \in \mathbf{W}^{1,r}(\widetilde{\mathcal{O}})$, we use the Hölder's inequality and the continuous embedding $\mathbf{H}^1_0(\widetilde{\mathcal{O}}) \hookrightarrow \mathbf{L}^{\frac{2r}{r-2}}(\widetilde{\mathcal{O}})$ to get:

$$\begin{aligned}
&\int_{\widetilde{\mathcal{O}}} \lambda^{-2} \left|(\widetilde{\boldsymbol{v}} \cdot \nabla)\, \widetilde{\boldsymbol{z}}_2\right|^2 e^{2se^{\lambda\psi}} \mathrm{d}x \leq \int_{\widetilde{\mathcal{O}}} |\nabla \widetilde{\boldsymbol{z}}_2|^2 \left|\lambda^{-1} \widetilde{\boldsymbol{v}} e^{se^{\lambda\psi}}\right|^2 \mathrm{d}x \\
&\quad \leq \ \|\nabla \widetilde{\boldsymbol{z}}_2\|^2_{\mathrm{L}^r(\widetilde{\mathcal{O}})} \left\|\lambda^{-1} \widetilde{\boldsymbol{v}} e^{se^{\lambda\psi}}\right\|^2_{\mathbf{L}^{\frac{2r}{r-2}}(\widetilde{\mathcal{O}})} \leq C \,\|\nabla \widetilde{\boldsymbol{z}}_2\|^2_{\mathrm{L}^r(\widetilde{\mathcal{O}})} \left\|\nabla(\lambda^{-1} \widetilde{\boldsymbol{v}} e^{se^{\lambda\psi}})\right\|^2_{\mathrm{L}^2(\widetilde{\mathcal{O}})} \\
&\quad \leq \ C \,\|\nabla \widetilde{\boldsymbol{z}}_2\|^2_{\mathrm{L}^r(\widetilde{\mathcal{O}})} \lambda^{-2} \Big(\int_{\widetilde{\mathcal{O}}} |\nabla \widetilde{\boldsymbol{v}}|^2 e^{2se^{\lambda\psi}} \mathrm{d}x + \int_{\widetilde{\mathcal{O}}} |\widetilde{\boldsymbol{v}}|^2 s^2 \lambda^2 e^{2\lambda\psi} e^{2se^{\lambda\psi}} \mathrm{d}x \Big).
\end{aligned} \tag{2.19}$$

Thus, gathering (2.17), (2.18) and (2.19) and choosing $\lambda \geq \widetilde{\mathfrak{m}}(\boldsymbol{z}_1, \boldsymbol{z}_2)\,\widehat{c}$ for $\widehat{c}$ large enough (and depending only on the geometry), the terms in $\widetilde{\boldsymbol{z}}_1$, $\widetilde{\boldsymbol{z}}_2$ at the right hand side of inequality (2.17) can be absorbed and we obtain

$$\begin{aligned}
&\int_{\widetilde{\mathcal{O}}} \Big(|\nabla \widetilde{\boldsymbol{v}}|^2 + se^{\lambda\psi} |\operatorname{curl} \widetilde{\boldsymbol{v}}|^2 + s^2 \lambda^2 e^{2\lambda\psi} |\widetilde{\boldsymbol{v}}|^2 \Big) e^{2se^{\lambda\psi}} \mathrm{d}x \\
&\quad \leq C \left(\int_{\widetilde{\mathcal{O}}} (s^{-1} \lambda^{-2} e^{-\lambda\psi} |\nabla \widetilde{d}|^2 + \lambda^{-2} |\widetilde{\boldsymbol{f}}|^2) e^{2se^{\lambda\psi}} \mathrm{d}x + s^3 \lambda^2 \int_\omega e^{3\lambda\psi} |\widetilde{\boldsymbol{v}}|^2 e^{2se^{\lambda\psi}} \mathrm{d}x \right).
\end{aligned} \tag{2.20}$$

Moreover,

$$\begin{aligned}
&\int_{\widetilde{\mathcal{O}} \backslash \overline{\mathcal{O}}} (s^{-1} \lambda^{-2} e^{-\lambda\psi} |\nabla \widetilde{d}|^2 + \lambda^{-2} |\widetilde{\boldsymbol{f}}|^2) e^{2se^{\lambda\psi}} \mathrm{d}x \\
&\qquad \leq \lambda^{-2} e^{2se^{\lambda c_0}} \int_{\widetilde{\mathcal{O}} \backslash \overline{\mathcal{O}}} (|\nabla \widetilde{d}|^2 + |\widetilde{\boldsymbol{f}}|^2) \mathrm{d}x \leq C \lambda^{-2} \widetilde{\mathfrak{m}}(\boldsymbol{z}_1, \boldsymbol{z}_2)^2 \left(\|\boldsymbol{v}\|^2_{\mathbf{H}^2(\mathcal{O})} + \|p\|^2_{\mathrm{H}^1(\mathcal{O})} \right).
\end{aligned}$$

In above calculations, we have used the fact that $\psi \leq c_0$ in $\widetilde{\mathcal{O}} \backslash \overline{\mathcal{O}}$ and (2.16). Then (2.12) follows by combining the above inequality with (2.20).

Finally, to prove (2.13), we first apply (2.3) to $(\widetilde{\boldsymbol{v}}, \widetilde{p})$ which gives:

$$\begin{aligned}
&\int_{\widetilde{\mathcal{O}}} se^{\lambda\psi} |\operatorname{div} \widetilde{\boldsymbol{v}} - \widetilde{p}|^2 e^{2se^{\lambda\psi}} \mathrm{d}x \\
&\quad \leq C \left(\int_\omega se^{\lambda\psi} |\operatorname{div} \widetilde{\boldsymbol{v}} - \widetilde{p}|^2 e^{2se^{\lambda\psi}} \mathrm{d}x + \int_{\widetilde{\mathcal{O}}} \lambda^{-2} |\widetilde{\boldsymbol{f}} - (\widetilde{\boldsymbol{z}}_1 \cdot \nabla)\, \widetilde{\boldsymbol{v}} - (\widetilde{\boldsymbol{v}} \cdot \nabla)\, \widetilde{\boldsymbol{z}}_2|^2 e^{2se^{\lambda\psi}} \mathrm{d}x \right).
\end{aligned}$$

Then, using (2.18) and (2.19) to estimate the last above integral, we obtain for $\widehat{c}$ large enough and $\lambda \geq \widehat{c}\,\widetilde{\mathfrak{m}}(\boldsymbol{z}_1, \boldsymbol{z}_2)$,

$$\begin{aligned}
\int_{\widetilde{\mathcal{O}}} se^{\lambda\psi} |\operatorname{div} \widetilde{\boldsymbol{v}} - \widetilde{p}|^2 e^{2se^{\lambda\psi}} \mathrm{d}x \leq C &\left(\int_\omega se^{\lambda\psi} |\operatorname{div} \widetilde{\boldsymbol{v}} - \widetilde{p}|^2 e^{2se^{\lambda\psi}} \mathrm{d}x + \int_{\widetilde{\mathcal{O}}} \lambda^{-2} |\widetilde{\boldsymbol{f}}|^2 e^{2se^{\lambda\psi}} \mathrm{d}x \right) \\
&+ \int_{\widetilde{\mathcal{O}}} (|\widetilde{\nabla \boldsymbol{v}}|^2 + s^2 \lambda^2 e^{2\lambda\psi} |\widetilde{\boldsymbol{v}}|^2) e^{2se^{\lambda\psi}} \mathrm{d}x.
\end{aligned}$$

Hence, we use (2.20) and the rest of the proof is the same as for (2.12). $\square$

3 Stability estimates for Oseen and Navier-Stokes Equations

In this section we use the Carleman inequalities given in Theorem 2.3 to obtain several stability estimates for both distributed and boundary observation. In particular, we prove Theorems 1.1 and 1.2. We first prove a Hölder type interior estimates and a global log type estimates for a distributed observation. Then, we use an extension of the domain procedure to obtain a global log type estimates for a boundary observation.

We recall that Ω is a nonempty bounded open subset of $\mathbb{R}^N$ ($N = 2$ or $N = 3$) with a boundary $\partial\Omega$ of class C^2, that Γ_{obs} is a nonempty open subset of $\partial\Omega$ and that ω is a nonempty open subset of Ω. Moreover, $\boldsymbol{z}_1$, $\boldsymbol{z}_2$ are vector fields satisfying (1.2) and we use the following notation for the particular constant:

$$\mathfrak{m}(\boldsymbol{z}_1, \boldsymbol{z}_2) \stackrel{\text{def}}{=} \max\Big\{1\,,\, \|\boldsymbol{z}_1\|_{\mathbf{L}^\infty(\Omega)}\,,\, \|\nabla \boldsymbol{z}_2\|_{\mathrm{L}^r(\Omega)}\Big\}. \tag{3.1}$$

We also recall that $C > 0$ denotes a generic constant only depending on the geometry and in particular independent on s, λ, $\boldsymbol{z}_1$, $\boldsymbol{z}_2$.

3.1 Stability estimates with a distributed observation

3.1.1 A Hölder type interior estimate

Theorem 3.1. *Let Ω_0 be an open subset such that $\omega \Subset \Omega_0 \Subset \Omega$. There exist $\widehat{c} > 0$, $\widehat{s} > 1$ and $c_1^* > c_2^* > 0$ such that for all $\boldsymbol{z}_1, \boldsymbol{z}_2$ satisfying* (1.2)*, all $\lambda \geq \widehat{\lambda} \stackrel{def}{=} \mathfrak{m}(\boldsymbol{z}_1, \boldsymbol{z}_2)\widehat{c}$ and all $s \geq \widehat{s}$, every solution $(\boldsymbol{v}, p) \in \mathbf{H}^2(\Omega) \times \mathrm{H}^1(\Omega)$ of the Oseen equations* (1.1) *satisfies:*

$$\begin{aligned}\|\boldsymbol{v}\|_{\mathbf{L}^2(\Omega_0)} + \|\operatorname{curl} \boldsymbol{v}\|_{(\mathrm{L}^2(\Omega_0))^{2N-3}} \leq e^{se^{c_1^*\lambda}} \Big(\|\boldsymbol{f}\|_{\mathbf{L}^2(\Omega)} + \|d\|_{\mathrm{H}^1(\Omega)} + \|\boldsymbol{v}\|_{\mathbf{L}^2(\omega)}\Big) \\ + e^{-se^{c_2^*\lambda}} \left(\|\boldsymbol{v}\|_{\mathbf{H}^2(\Omega)} + \|p\|_{\mathrm{H}^1(\Omega)}\right)\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}\|p - \operatorname{div} \boldsymbol{v}\|_{\mathrm{L}^2(\Omega_0)} \leq e^{se^{c_1^*\lambda}} \Big(\|\boldsymbol{f}\|_{\mathbf{L}^2(\Omega)} + \|d\|_{\mathrm{H}^1(\Omega)} + \|\boldsymbol{v}\|_{\mathbf{L}^2(\omega)} + \|p\|_{\mathrm{L}^2(\omega)}\Big) \\ + e^{-se^{c_2^*\lambda}} \left(\|\boldsymbol{v}\|_{\mathbf{H}^2(\Omega)} + \|p\|_{\mathrm{H}^1(\Omega)}\right).\end{aligned} \tag{3.3}$$

Proof. Let us introduce $\psi_{\min} \stackrel{\text{def}}{=} \min\limits_{x\in\Omega_0} \psi(x)$ and $\psi_{\max} \stackrel{\text{def}}{=} \max\limits_{x\in\Omega} \psi(x)$. We apply (2.12) to $(\boldsymbol{v}, p)$ to get, with $\widehat{\lambda} \leq \lambda$,

$$\begin{aligned}&\int_\Omega \Big(s^2\lambda^2 e^{2\lambda\psi}|\boldsymbol{v}|^2 + se^{\lambda\psi}|\operatorname{curl} \boldsymbol{v}|^2\Big) e^{2se^{\lambda\psi}} \mathrm{d}x \\ &\qquad \leq C \left(\int_\Omega (|\boldsymbol{f}|^2 + |\nabla d|^2) e^{2se^{\lambda\psi_{\max}}} \mathrm{d}x + s^3\lambda^2 e^{3\lambda\psi_{\max}} e^{2se^{\lambda\psi_{\max}}} \|\boldsymbol{v}\|^2_{\mathbf{L}^2(\omega)} \right. \\ &\qquad\qquad \left. + e^{2se^{\lambda c_0}} \left(\|\boldsymbol{v}\|^2_{\mathbf{H}^2(\Omega)} + \|p\|^2_{\mathrm{H}^1(\Omega)}\right)\right)\end{aligned} \tag{3.4}$$

and then,

$$\begin{aligned}\int_{\Omega_0} \Big(s^2\lambda^2 e^{2\lambda\psi_{\min}}|\boldsymbol{v}|^2 + se^{\lambda\psi_{\min}}|\operatorname{curl} \boldsymbol{v}|^2\Big) e^{2se^{\lambda\psi_{\min}}} \mathrm{d}x \leq C \left(\int_\Omega (|\boldsymbol{f}|^2 + |\nabla d|^2) e^{2se^{\lambda\psi_{\max}}} \mathrm{d}x \right. \\ \left. + s^3\lambda^2 e^{3\lambda\psi_{\max}} e^{2se^{\lambda\psi_{\max}}} \|\boldsymbol{v}\|^2_{\mathbf{L}^2(\omega)} + e^{2se^{\lambda c_0}} \left(\|\boldsymbol{v}\|^2_{\mathbf{H}^2(\Omega)} + \|p\|^2_{\mathrm{H}^1(\Omega)}\right)\right).\end{aligned}$$

Thus, by dividing the above inequality by $e^{2se^{\lambda\psi_{\min}}}$ we obtain (3.2) for some $c_1^* > c_2^* > 0$ independent on λ. Estimate (3.3) is obtained analogously. □

As a consequence of Theorem 3.1, we have the following

Theorem 3.2. *Let* Ω_0 *be an open subset such that* $\omega \Subset \Omega_0 \Subset \Omega$. *There exists* $c^* > 0$ *such that for all* $\boldsymbol{z}_1, \boldsymbol{z}_2$ *satisfying* (1.2), *all* $\lambda \geq \widehat{\lambda} \overset{def}{=} \mathfrak{m}(\boldsymbol{z}_1, \boldsymbol{z}_2)\widehat{c}$, *there exists* $\beta \in (0, 1/2)$ *such that every solution* $(\boldsymbol{v}, p) \in \mathbf{H}^2(\Omega) \times \mathrm{H}^1(\Omega)$ *of the Oseen equations* (1.1) *satisfies:*

$$\begin{aligned}&\|\boldsymbol{v}\|_{\mathbf{L}^2(\Omega_0)} + \|\operatorname{curl} \boldsymbol{v}\|_{\left(\mathbf{L}^2(\Omega_0)\right)^{2N-3}} \\ &\qquad\leq e^{e^{c^*\lambda}} \left(\|\boldsymbol{f}\|_{\mathbf{L}^2(\Omega)} + \|d\|_{\mathrm{H}^1(\Omega)} + \|\boldsymbol{v}\|_{\mathbf{L}^2(\omega)}\right)^{\beta} \left(\|\boldsymbol{v}\|_{\mathbf{H}^2(\Omega)} + \|p\|_{\mathrm{H}^1(\Omega)}\right)^{1-\beta}\end{aligned}$$

and

$$\begin{aligned}&\|p - \operatorname{div} \boldsymbol{v}\|_{\mathrm{L}^2(\Omega_0)} \\ &\quad\leq e^{e^{c^*\lambda}} \left(\|\boldsymbol{f}\|_{\mathbf{L}^2(\Omega)} + \|d\|_{\mathrm{H}^1(\Omega)} + \|\boldsymbol{v}\|_{\mathbf{L}^2(\omega)} + \|p\|_{\mathrm{L}^2(\omega)}\right)^{\beta} \left(\|\boldsymbol{v}\|_{\mathbf{H}^2(\Omega)} + \|p\|_{\mathrm{H}^1(\Omega)}\right)^{1-\beta}.\end{aligned}$$

Proof. Since the proofs are analogous we only prove the first inequality. For that, we apply Theorem 3.1 and, for $s > \widehat{s}$, Inequality (3.2) rewrites $\|\boldsymbol{v}\|_{\mathbf{L}^2(\Omega_0)} + \|\operatorname{curl} \boldsymbol{v}\|_{(\mathrm{L}^2(\Omega_0))^{2N-3}} \leq e^{sC_1^*}A + e^{-sC_2^*}B$ where $C_i^* \overset{\text{def}}{=} e^{c_i^*\lambda}$ for $i = 1, 2$. First, we suppose that $\frac{1}{C_1^*+C_2^*} \ln\left(\frac{B}{A}\right) \geq \widehat{s}$ and we choose $s = \frac{1}{C_1^*+C_2^*} \ln\left(\frac{B}{A}\right)$. Hence, we obtain

$$\|\boldsymbol{v}\|_{\mathbf{L}^2(\Omega_0)} + \|\operatorname{curl} \boldsymbol{v}\|_{\left(\mathbf{L}^2(\Omega_0)\right)^{2N-3}} \leq 2A^{\frac{C_2^*}{C_1^*+C_2^*}} B^{\frac{C_1^*}{C_1^*+C_2^*}}. \tag{3.5}$$

Secondly, if $\frac{1}{C_1^*+C_2^*} \ln\left(\frac{B}{A}\right) < \widehat{s}$, then $B < e^{(C_1^*+C_2^*)\widehat{s}}A$. Hence, we also obtain (3.5) using the existence of $C > 0$ such that $\|\boldsymbol{v}\|_{\mathbf{L}^2(\Omega_0)} + \|\operatorname{curl} \boldsymbol{v}\|_{\mathbf{L}^2(\Omega_0)} \leq CB$. ☐

Remark 3.3. *According to the proof of Theorem 3.2, we have* $\beta = \beta(\lambda) = \frac{e^{c_2^*\lambda}}{e^{c_1^*\lambda}+e^{c_2^*\lambda}}$ *where* $c_1^* > c_2^* > 0$ *are the constants given in Theorem 3.1 which only depend on the geometry. Therefore,* $\beta(\lambda) \in (0, 1/2)$ *and* $\beta(\lambda) \to 0$ *as* $\lambda \to +\infty$.

3.1.2 A global logarithmic estimate

Theorem 3.4. *There exist* $\widehat{c} > 0$, $\widehat{s} > 1$ *such that for all* $\boldsymbol{z}_1, \boldsymbol{z}_2$ *satisfying* (1.2), *all* $\lambda \geq \widehat{\lambda} \overset{def}{=} \mathfrak{m}(\boldsymbol{z}_1, \boldsymbol{z}_2)\widehat{c}$ *and all* $s \geq \widehat{s}$, *every solution* $(\boldsymbol{v}, p) \in \mathbf{H}^2(\Omega) \times \mathrm{H}^1(\Omega)$ *of the Oseen equations* (1.1) *satisfies:*

$$\|\boldsymbol{v}\|_{\mathbf{L}^2(\Omega)} \leq e^{se^{c^*\lambda}} \left(\|\boldsymbol{f}\|_{\mathbf{L}^2(\Omega)} + \|d\|_{\mathrm{H}^1(\Omega)} + \|\boldsymbol{v}\|_{\mathbf{L}^2(\omega)}\right) + \frac{1}{s}\left(\|\boldsymbol{v}\|_{\mathbf{H}^2(\Omega)} + \|p\|_{\mathrm{H}^1(\Omega)}\right), \tag{3.6}$$

$$\|\operatorname{curl} \boldsymbol{v}\|_{(\mathrm{L}^2(\Omega))^{2N-3}} \leq e^{se^{c^*\lambda}} \left(\|\boldsymbol{f}\|_{\mathbf{L}^2(\Omega)} + \|d\|_{\mathrm{H}^1(\Omega)} + \|\boldsymbol{v}\|_{\mathbf{L}^2(\omega)}\right) + \frac{1}{s^{1/2}}\left(\|\boldsymbol{v}\|_{\mathbf{H}^2(\Omega)} + \|p\|_{\mathrm{H}^1(\Omega)}\right) \tag{3.7}$$

and

$$\begin{aligned}\|p - \operatorname{div} \boldsymbol{v}\|_{\mathrm{L}^2(\Omega)} \leq e^{se^{c^*\lambda}} &\left(\|\boldsymbol{f}\|_{\mathbf{L}^2(\Omega)} + \|d\|_{\mathrm{H}^1(\Omega)} + \|\boldsymbol{v}\|_{\mathbf{L}^2(\omega)} + \|p\|_{\mathrm{L}^2(\omega)}\right) \\ &+ \frac{1}{s^{1/2}}\left(\|\boldsymbol{v}\|_{\mathbf{H}^2(\Omega)} + \|p\|_{\mathrm{H}^1(\Omega)}\right).\end{aligned} \tag{3.8}$$

Proof. We apply (2.12) to $(\boldsymbol{v}, p)$, with $\widehat{\lambda} \leq \lambda$, to get (3.4) as in Theorem 3.1. Thus, by dividing inequality (3.4) by $e^{2se^{\lambda c_0}}$ and using the fact that

$$e^{-2se^{\lambda c_0}} \int_\Omega e^{2\lambda\psi} |\boldsymbol{v}|^2 e^{2se^{\lambda\psi}} \mathrm{d}x \geq \int_\Omega |\boldsymbol{v}|^2 \mathrm{d}x$$

we obtain (3.6) and (3.7) for some $c^* > 0$ large enough (independent on λ). Proceeding as previously (but with (2.13) instead of (2.12)) we obtain (3.8). $\square$

Then, we deduce the following

Theorem 3.5. *There exist $\widehat{c} > 0$ and $c^* > 0$ such that for all $\boldsymbol{z}_1, \boldsymbol{z}_2$ satisfying* (1.2)*, all $\lambda \geq \widehat{\lambda} \overset{def}{=} \mathfrak{m}(\boldsymbol{z}_1, \boldsymbol{z}_2)\, \widehat{c}$, every solution $(\boldsymbol{v}, p) \in \mathbf{H}^2(\Omega) \times \mathrm{H}^1(\Omega)$ of the Oseen equations* (1.1) *satisfies:*

$$\|\boldsymbol{v}\|_{\mathbf{L}^2(\Omega)} \leq \frac{e^{e^{c^*\lambda}} \left(\|\boldsymbol{v}\|_{\mathbf{H}^2(\Omega)} + \|p\|_{\mathrm{H}^1(\Omega)} \right)}{\ln\left(1 + \dfrac{\|\boldsymbol{v}\|_{\mathbf{H}^2(\Omega)} + \|p\|_{\mathrm{H}^1(\Omega)}}{\|\boldsymbol{f}\|_{\mathbf{L}^2(\Omega)} + \|d\|_{\mathrm{H}^1(\Omega)} + \|\boldsymbol{v}\|_{\mathbf{L}^2(\omega)}}\right)}, \tag{3.9}$$

$$\|\mathrm{curl}\, \boldsymbol{v}\|_{(\mathrm{L}^2(\Omega))^{2N-3}} \leq \frac{e^{e^{c^*\lambda}} \left(\|\boldsymbol{v}\|_{\mathbf{H}^2(\Omega)} + \|p\|_{\mathrm{H}^1(\Omega)} \right)}{\left(\ln\left(1 + \dfrac{\|\boldsymbol{v}\|_{\mathbf{H}^2(\Omega)} + \|p\|_{\mathrm{H}^1(\Omega)}}{\|\boldsymbol{f}\|_{\mathbf{L}^2(\Omega)} + \|d\|_{\mathrm{H}^1(\Omega)} + \|\boldsymbol{v}\|_{\mathbf{L}^2(\omega)}}\right)\right)^{1/2}} \tag{3.10}$$

and

$$\|p - \mathrm{div}\, \boldsymbol{v}\|_{\mathrm{L}^2(\Omega)} \leq \frac{e^{e^{c^*\lambda}} \left(\|\boldsymbol{v}\|_{\mathbf{H}^2(\Omega)} + \|p\|_{\mathrm{H}^1(\Omega)} \right)}{\left(\ln\left(1 + \dfrac{\|\boldsymbol{v}\|_{\mathbf{H}^2(\Omega)} + \|p\|_{\mathrm{H}^1(\Omega)}}{\|\boldsymbol{f}\|_{\mathbf{L}^2(\Omega)} + \|d\|_{\mathrm{H}^1(\Omega)} + \|\boldsymbol{v}\|_{\mathbf{L}^2(\omega)} + \|p\|_{\mathrm{L}^2(\omega)}}\right)\right)^{1/2}}. \tag{3.11}$$

Proof. We apply Theorem 3.4 and, for $s > \hat{s}$, we introduce A and B such that we rewrite (3.6) as $\|\boldsymbol{v}\|_{\mathbf{L}^2(\Omega)} \leq e^{sC^*} A + \frac{c^*}{s} B$ where $C^* \overset{\mathrm{def}}{=} e^{c^*\lambda}$. First, we suppose that $\frac{1}{2C^*} \ln(1 + \frac{B}{A}) \geq \widehat{s}$ and we choose $s = \frac{1}{2C^*} \ln(1 + \frac{B}{A})$. It yields

$$\|\boldsymbol{v}\|_{\mathbf{L}^2(\Omega)} \leq B \left(\left(1 + \frac{B}{A}\right)^{1/2} \frac{A}{B} + \frac{2C^* c^*}{\ln(1 + \frac{B}{A}))} \right)$$

and next, using the fact that $\frac{1}{x} \leq \frac{1}{\ln(1+x)}$ if $0 < x < 1$, *i.e.* $B \leq A$ and $\frac{1}{x^{1/2}} \leq \frac{1}{\ln(1+x)}$ if $x > 1$, *i.e.* $B > A$, we obtain (3.9) (by choosing $c^* > 0$ larger if necessary).

In the case $\frac{1}{2C^*} \ln(1 + \frac{B}{A}) \leq \widehat{s}$ we have $B \leq e^{e^{c^*\lambda}} A$ for some (other) constant $c^* > 0$ and (3.6) with $s = \widehat{s}$ gives $\|\boldsymbol{v}\|_{\mathbf{L}^2(\Omega)} \leq e^{e^{c^*\lambda}} A$ for some (other) constant $c^* > 0$. Then the conclusion follows from $A = B\frac{A}{B} \leq B\frac{1}{\ln(1+\frac{B}{A})}$ (since $\frac{1}{x} \leq \frac{1}{\ln(1+x)}$ for all $x > 0$).

The proof of (3.10) and (3.11) are obtained in a similar way. $\square$

3.2 Stability estimates with boundary observation

We now prove the following theorem from which we deduce the logarithm estimates stated in Theorem 1.1 as in the proof of Theorem 3.5. Notice that the first estimate (1.4) is given in the previous Theorem 3.5 (see (3.9)).

Theorem 3.6. *There exists* $\widehat{c} > 0$, $\widehat{s} > 1$ *such that for all* $\boldsymbol{z}_1, \boldsymbol{z}_2$ *satisfying* (1.2)*, all* $\lambda \geq \widehat{\lambda} \overset{def}{=} \mathfrak{m}(\boldsymbol{z}_1, \boldsymbol{z}_2)\widehat{c}$ *and all* $s \geq \widehat{s}$*, every solution* $(\boldsymbol{v}, p) \in \mathbf{H}^2(\Omega) \times \mathrm{H}^1(\Omega)$ *of the Oseen equations* (1.1) *satisfies:*

$$\|\boldsymbol{v}\|_{\mathbf{L}^2(\Omega)} \leq e^{se^{c^*\lambda}} \Big(\|\boldsymbol{f}\|_{\mathbf{L}^2(\Omega)} + \|d\|_{\mathrm{H}^1(\Omega)} + \|\boldsymbol{v}\|_{\mathbf{H}^{3/2}(\Gamma_{\mathrm{obs}})} + \|\sigma(\boldsymbol{v}, p)\mathbf{n}\|_{\mathbf{H}^{1/2}(\Gamma_{\mathrm{obs}})} \Big) + \frac{1}{s} \left(\|\boldsymbol{v}\|_{\mathbf{H}^2(\Omega)} + \|p\|_{\mathrm{H}^1(\Omega)} \right) \tag{3.12}$$

and

$$\begin{aligned} &\|\operatorname{curl} \boldsymbol{v}\|_{(\mathrm{L}^2(\Omega))^{2N-3}} + \|p - \operatorname{div} \boldsymbol{v}\|_{\mathrm{L}^2(\Omega)} \\ &\qquad \leq e^{se^{c^*\lambda}} \Big(\|\boldsymbol{f}\|_{\mathbf{L}^2(\Omega)} + \|d\|_{\mathrm{H}^1(\Omega)} + \|\boldsymbol{v}\|_{\mathbf{H}^{3/2}(\Gamma_{\mathrm{obs}})} + \|\sigma(\boldsymbol{v}, p)\mathbf{n}\|_{\mathbf{H}^{1/2}(\Gamma_{\mathrm{obs}})} \Big) \\ &\qquad\qquad + \frac{1}{s^{1/2}} \left(\|\boldsymbol{v}\|_{\mathbf{H}^2(\Omega)} + \|p\|_{\mathrm{H}^1(\Omega)} \right). \end{aligned} \tag{3.13}$$

Let us begin by proving the following lemma which is a construction of an extension of the domain Ω and of the solution $(\boldsymbol{v}, p)$ of Problem (1.1):

Lemma 3.7. *Let* $\widetilde{\Omega}$ *be an extension of* Ω *of class* C^2 *through* Γ_{obs} *(see Figure 1), namely*

$$\widetilde{\Omega} \text{ is of class } C^2, \quad \partial\Omega \cap \widetilde{\Omega} = \Gamma_{\mathrm{obs}}.$$

There exists an extension $(\widetilde{\boldsymbol{v}}, \widetilde{p}) \in \mathbf{H}^2(\widetilde{\Omega}) \times \mathrm{H}^1(\widetilde{\Omega})$ *of* $(\boldsymbol{v}, p) \in \mathbf{H}^2(\Omega) \times \mathrm{H}^1(\Omega)$ *such that*

$$\widetilde{\boldsymbol{v}}_{|\Gamma_{\mathrm{obs}}} = \boldsymbol{v}_{|\Gamma_{\mathrm{obs}}}, \quad \tfrac{\partial \widetilde{\boldsymbol{v}}}{\partial \mathbf{n}}\big|_{\Gamma_{\mathrm{obs}}} = \tfrac{\partial \boldsymbol{v}}{\partial \mathbf{n}}\big|_{\Gamma_{\mathrm{obs}}}, \quad \widetilde{p}_{|\Gamma_{\mathrm{obs}}} = p_{|\Gamma_{\mathrm{obs}}}$$

with the following estimate

$$\|\widetilde{\boldsymbol{v}}\|^2_{\mathbf{H}^2(\widetilde{\Omega}\backslash\overline{\Omega})} + \|\widetilde{p}\|^2_{\mathrm{H}^1(\widetilde{\Omega}\backslash\overline{\Omega})} \leq C \left(\|\boldsymbol{v}\|^2_{\mathbf{H}^{3/2}(\Gamma_{\mathrm{obs}})} + \left\| \frac{\partial \boldsymbol{v}}{\partial \mathbf{n}} \right\|^2_{\mathbf{H}^{1/2}(\Gamma_{\mathrm{obs}})} + \|p\|^2_{\mathrm{H}^{1/2}(\Gamma_{\mathrm{obs}})} \right). \tag{3.14}$$

In particular,

$$\|\widetilde{\boldsymbol{v}}\|^2_{\mathbf{H}^2(\widetilde{\Omega})} + \|\widetilde{p}\|^2_{\mathrm{H}^1(\widetilde{\Omega})} \leq C \left(\|\boldsymbol{v}\|^2_{\mathbf{H}^2(\Omega)} + \|p\|^2_{\mathrm{H}^1(\Omega)} \right). \tag{3.15}$$

Proof. We consider the linear continuous trace-right inverse operator (see for example [19, Theorem 1.5.1.2])

$$\begin{array}{rrcl} R: & \mathbf{H}^{3/2}(\Gamma_{\mathrm{obs}}) \times \mathbf{H}^{1/2}(\Gamma_{\mathrm{obs}}) \times \mathrm{H}^{1/2}(\Gamma_{\mathrm{obs}}) & \longrightarrow & \mathbf{H}^2(\Omega) \times \mathrm{H}^1(\Omega) \\ & (\boldsymbol{g}_{\mathrm{obs}}, \boldsymbol{h}_{\mathrm{obs}}, k_{\mathrm{obs}}) & \longmapsto & (\boldsymbol{w}, q) \end{array}$$

with $(\boldsymbol{w}, \frac{\partial \boldsymbol{w}}{\partial \mathbf{n}}, q) = (\boldsymbol{g}_{\mathrm{obs}}, \boldsymbol{h}_{\mathrm{obs}}, k_{\mathrm{obs}})$ on Γ_{obs}. Then, let us denote by S the linear continuous extension operator given by Stein's theorem (see [1]):

$$\begin{array}{rrcl} S: & \mathbf{H}^2(\Omega) \times \mathrm{H}^1(\Omega) & \longrightarrow & \mathbf{H}^2(\mathbb{R}^N) \times \mathrm{H}^1(\mathbb{R}^N) \ . \\ & (\boldsymbol{w}, q) & \longmapsto & (\boldsymbol{W}, Q) \end{array}$$

We also denote by T the linear continuous operator of restriction to $\widetilde{\Omega}$:

$$\begin{array}{rrcl} T: & \mathbf{H}^2(\mathbb{R}^N) \times \mathrm{H}^1(\mathbb{R}^N) & \longrightarrow & \mathbf{H}^2(\widetilde{\Omega}) \times \mathrm{H}^1(\widetilde{\Omega}) \ . \\ & (\boldsymbol{W}, Q) & \longmapsto & (\boldsymbol{W}_{|\widetilde{\Omega}}, Q_{|\widetilde{\Omega}}) \end{array}$$

Finally, by denoting $(\widetilde{\boldsymbol{w}}, \widetilde{q}) \overset{\mathrm{def}}{=} T \circ S \circ R\,(\boldsymbol{v}, \partial_{\mathrm{n}}\boldsymbol{v}, p)$, we conclude by defining

$$\widetilde{\boldsymbol{v}} \overset{\mathrm{def}}{=} \begin{cases} \boldsymbol{v} & \text{in } \Omega \\ \widetilde{\boldsymbol{w}} & \text{in } \widetilde{\Omega}\backslash\overline{\Omega} \end{cases} \qquad \text{and} \qquad \widetilde{q} \overset{\mathrm{def}}{=} \begin{cases} p & \text{in } \Omega \\ \widetilde{q} & \text{in } \widetilde{\Omega}\backslash\overline{\Omega}. \end{cases}$$

It is easily checked that $(\widetilde{\boldsymbol{v}}, \widetilde{p}) \in \mathbf{H}^2(\widetilde{\Omega}) \times \mathrm{H}^1(\widetilde{\Omega})$. □

Proof of Theorem 3.6. In what follows, $\widetilde{\boldsymbol{z}}_1$, $\widetilde{\boldsymbol{z}}_2$ denote some continuous extensions to $\mathbb{R}^N$ of $\boldsymbol{z}_1$, $\boldsymbol{z}_2$, for the $\mathbf{L}^\infty$ and the $\mathbf{W}^{1,r}$ norm respectively. Let us consider the extensions $\widetilde{\Omega}$ and $(\widetilde{\boldsymbol{v}}, \widetilde{q}) \in \mathbf{H}^2(\widetilde{\Omega}) \times \mathrm{H}^1(\widetilde{\Omega})$ given by Lemma 3.7. Let us consider $\omega \Subset \widetilde{\Omega} \backslash \overline{\Omega}$ a non empty bounded open subset. We summarize these notations in Figure 1.

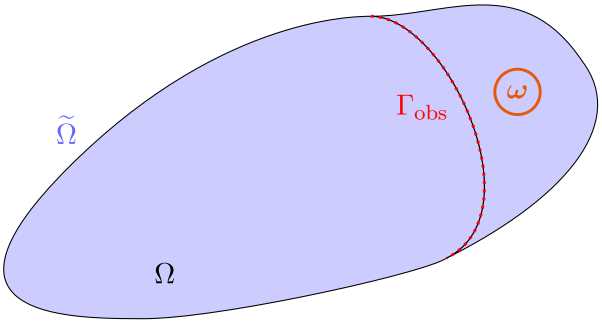

Figure 1: Notations

Next, we apply (2.12) and (2.13) to $(\widetilde{\boldsymbol{v}}, \widetilde{p})$ and, with $\widehat{\lambda} \leq \lambda$, we get:

$$\begin{aligned}
&\int_{\widetilde{\Omega}} \left(s^2\lambda^2 e^{2\lambda\psi} |\widetilde{\boldsymbol{v}}|^2 + s e^{\lambda\psi} |\operatorname{curl} \widetilde{\boldsymbol{v}}|^2 + s e^{\lambda\psi} |\widetilde{p} - \operatorname{div} \widetilde{\boldsymbol{v}}|^2 \right) e^{2se^{\lambda\psi}} \mathrm{d}x \\
&\qquad \leq C \left(\int_{\widetilde{\Omega}} (| - \nu \Delta \widetilde{\boldsymbol{v}} + (\widetilde{\boldsymbol{z}}_1 \cdot \nabla)\, \widetilde{\boldsymbol{v}} + (\widetilde{\boldsymbol{v}} \cdot \nabla)\, \widetilde{\boldsymbol{z}}_2 + \nabla \widetilde{p}|^2 + |\nabla \operatorname{div} \widetilde{\boldsymbol{v}}|^2) e^{2se^{\lambda\psi}} \mathrm{d}x \right. \\
&\qquad\quad \left. + \int_{\omega} (s^3\lambda^2 e^{3\lambda\psi} |\widetilde{\boldsymbol{v}}|^2 + s e^{\lambda\psi} |\widetilde{p} - \operatorname{div} \widetilde{\boldsymbol{v}}|^2) e^{2se^{\lambda\psi}} \mathrm{d}x + e^{2se^{\lambda c_0}} \left(\|\boldsymbol{v}\|^2_{\mathbf{H}^2(\widetilde{\Omega})} + \|p\|^2_{\mathrm{H}^1(\widetilde{\Omega})} \right) \right),
\end{aligned}$$

and from (3.14), (3.15) and $\omega \subset \widetilde{\Omega} \backslash \overline{\Omega}$ we deduce

$$\begin{aligned}
&\int_{\Omega} \left(s^2\lambda^2 e^{2\lambda\psi} |\boldsymbol{v}|^2 + s e^{\lambda\psi} |\operatorname{curl} \boldsymbol{v}|^2 + s e^{\lambda\psi} |p - \operatorname{div} \boldsymbol{v}|^2 \right) e^{2se^{\lambda\psi}} \mathrm{d}x \\
&\leq \quad C \Bigg(\int_{\Omega} \left(|\nabla d|^2 + |\boldsymbol{f}|^2 \right) e^{2se^{\lambda\psi}} \mathrm{d}x + e^{3\lambda\psi_{\max}} e^{2se^{\lambda\psi_{\max}}} \int_{\widetilde{\Omega} \backslash \overline{\Omega}} \Big(s^3\lambda^2 |\widetilde{\boldsymbol{v}}|^2 \\
&\qquad\quad + s|\widetilde{p} - \operatorname{div} \widetilde{\boldsymbol{v}}|^2 + |\Delta \widetilde{\boldsymbol{v}}|^2 + |(\widetilde{\boldsymbol{z}}_1 \cdot \nabla)\, \widetilde{\boldsymbol{v}}|^2 + |(\widetilde{\boldsymbol{v}} \cdot \nabla)\, \widetilde{\boldsymbol{z}}_2|^2 + |\nabla \widetilde{p}|^2 + |\nabla \operatorname{div} \widetilde{\boldsymbol{v}}|^2 \Big) \mathrm{d}x \\
&\qquad\quad + e^{2se^{\lambda c_0}} \left(\|\boldsymbol{v}\|^2_{\mathbf{H}^2(\Omega)} + \|p\|^2_{\mathrm{H}^1(\Omega)} \right) \Bigg) \\
&\leq \quad C \Bigg(s^3\lambda^2 e^{3\lambda\psi_{\max}} e^{2se^{\lambda\psi_{\max}}} \mathfrak{m}(\boldsymbol{z}_1, \boldsymbol{z}_2)^2 \left(\|\boldsymbol{v}\|^2_{\mathbf{H}^{3/2}(\Gamma_{\text{obs}})} + \left\| \frac{\partial \boldsymbol{v}}{\partial \mathbf{n}} \right\|^2_{\mathbf{H}^{1/2}(\Gamma_{\text{obs}})} + \|p\|^2_{\mathbf{H}^{1/2}(\Gamma_{\text{obs}})} \right) \\
&\qquad\quad \int_{\Omega} \left(|\nabla d|^2 + |\boldsymbol{f}|^2 \right) e^{2se^{\lambda\psi}} \mathrm{d}x + e^{2se^{\lambda c_0}} \left(\|\boldsymbol{v}\|^2_{\mathbf{H}^2(\Omega)} + \|p\|^2_{\mathrm{H}^1(\Omega)} \right) \Bigg).
\end{aligned}$$

Here, we have used the notation $\psi_{\max} \overset{\text{def}}{=} \max_{x \in \Omega} \psi(x)$. Thus, by dividing the above inequality by $e^{2se^{\lambda c_0}}$ and using that

$$e^{-2se^{\lambda c_0}} \int_{\Omega} e^{2\lambda\psi} |\boldsymbol{v}|^2 e^{2se^{\lambda\psi}} \mathrm{d}x \geq \int_{\Omega} |\boldsymbol{v}|^2 \mathrm{d}x$$

and that $\lambda \geq \mathfrak{m}(\boldsymbol{z}_1, \boldsymbol{z}_2)\widehat{c}$, we obtain for some $c^* > 0$ large enough (independent on λ),

$$\|\boldsymbol{v}\|^2_{\mathbf{L}^2(\Omega)} \leq e^{se^{c^*\lambda}} \left(\|\boldsymbol{f}\|^2_{\mathbf{L}^2(\Omega)} + \|d\|^2_{\mathrm{H}^1(\Omega)} + \|\boldsymbol{v}\|^2_{\mathbf{H}^{3/2}(\Gamma_{\mathrm{obs}})} + \left\|\frac{\partial \boldsymbol{v}}{\partial \mathbf{n}}\right\|^2_{\mathbf{H}^{1/2}(\Gamma_{\mathrm{obs}})} + \|p\|^2_{\mathrm{H}^{1/2}(\Gamma_{\mathrm{obs}})} \right) + \frac{1}{s^2}\left(\|\boldsymbol{v}\|^2_{\mathbf{H}^2(\Omega)} + \|p\|^2_{\mathrm{H}^1(\Omega)} \right). \quad (3.16)$$

With a similar argument,

$$\|\operatorname{curl} \boldsymbol{v}\|^2_{(\mathrm{L}^2(\Omega))^{2N-3}} + \|p - \operatorname{div} \boldsymbol{v}\|^2_{\mathrm{L}^2(\Omega)} \leq e^{se^{c^*\lambda}} \left(\|\boldsymbol{f}\|^2_{\mathbf{L}^2(\Omega)} + \|d\|^2_{\mathrm{H}^1(\Omega)} + \|\boldsymbol{v}\|^2_{\mathbf{H}^{3/2}(\Gamma_{\mathrm{obs}})} + \left\|\frac{\partial \boldsymbol{v}}{\partial \mathbf{n}}\right\|^2_{\mathbf{H}^{1/2}(\Gamma_{\mathrm{obs}})} + \|p\|^2_{\mathrm{H}^{1/2}(\Gamma_{\mathrm{obs}})} \right) + \frac{1}{s}\left(\|\boldsymbol{v}\|^2_{\mathbf{H}^2(\Omega)} + \|p\|^2_{\mathrm{H}^1(\Omega)} \right). \quad (3.17)$$

Now, to conclude, it remains to replace the term $\|\frac{\partial \boldsymbol{v}}{\partial \mathbf{n}}\|^2_{\mathbf{H}^{1/2}(\Gamma_{\mathrm{obs}})} + \|p\|^2_{\mathrm{H}^{1/2}(\Gamma_{\mathrm{obs}})}$ by $\|\sigma(\boldsymbol{v}, p)\mathbf{n}\|^2_{\mathbf{H}^{1/2}(\Gamma_{\mathrm{obs}})}$. First, from

$$d = \operatorname{div} \boldsymbol{v} = \frac{\partial \boldsymbol{v}}{\partial \mathbf{n}} \cdot \mathbf{n} + \operatorname{div}_\tau \boldsymbol{v}_\tau + (\operatorname{div} \mathbf{n})(\boldsymbol{v} \cdot \mathbf{n}) \quad \text{on} \quad \Gamma_{\mathrm{obs}},$$

we deduce that

$$\left\|\frac{\partial \boldsymbol{v}}{\partial \mathbf{n}} \cdot \mathbf{n}\right\|^2_{\mathrm{H}^{1/2}(\Gamma_{\mathrm{obs}})} \leq C\left(\|\boldsymbol{v}\|^2_{\mathbf{H}^{3/2}(\Gamma_{\mathrm{obs}})} + \|d\|^2_{\mathrm{H}^1(\Omega)} \right). \quad (3.18)$$

The above inequality with the following computations

$$\begin{aligned} \nu \frac{\partial \boldsymbol{v}}{\partial \mathbf{n}} &= \sigma(\boldsymbol{v}, p)\mathbf{n} + p\mathbf{n} - \nu\, {}^t\nabla \boldsymbol{v} \mathbf{n} \\ &= \sigma(\boldsymbol{v}, p)\mathbf{n} + p\mathbf{n} - \nu \nabla(\boldsymbol{v} \cdot \mathbf{n}) + \nu(\nabla \mathbf{n})\, \boldsymbol{v} \\ &= \sigma(\boldsymbol{v}, p)\mathbf{n} + p\mathbf{n} - \nu \left(\frac{\partial \boldsymbol{v}}{\partial \mathbf{n}} \cdot \mathbf{n} \right) \mathbf{n} - \nu \nabla_\tau(\boldsymbol{v} \cdot \mathbf{n}) + \nu(\nabla \mathbf{n})\, \boldsymbol{v}, \end{aligned}$$

yields

$$\left\|\frac{\partial \boldsymbol{v}}{\partial \mathbf{n}}\right\|^2_{\mathbf{H}^{1/2}(\Gamma_{\mathrm{obs}})} \leq C\left(\|\sigma(\boldsymbol{v}, p)\mathbf{n}\|^2_{\mathbf{H}^{1/2}(\Gamma_{\mathrm{obs}})} + \|p\|^2_{\mathrm{H}^{1/2}(\Gamma_{\mathrm{obs}})} + \|\boldsymbol{v}\|^2_{\mathbf{H}^{3/2}(\Gamma_{\mathrm{obs}})} + \|d\|^2_{\mathrm{H}^1(\Omega)} \right).$$

Finally, from $p = 2\nu \frac{\partial \boldsymbol{v}}{\partial \mathbf{n}} \cdot \mathbf{n} - \sigma(\boldsymbol{v}, p)\mathbf{n} \cdot \mathbf{n}$ and (3.18) we deduce that

$$\|p\|^2_{\mathrm{H}^{1/2}(\Gamma_{\mathrm{obs}})} \leq C\left(\|\sigma(\boldsymbol{v}, p)\mathbf{n}\|^2_{\mathbf{H}^{1/2}(\Gamma_{\mathrm{obs}})} + \|\boldsymbol{v}\|^2_{\mathbf{H}^{3/2}(\Gamma_{\mathrm{obs}})} + \|d\|^2_{\mathrm{H}^1(\Omega)} \right)$$

and then

$$\left\|\frac{\partial \boldsymbol{v}}{\partial \mathbf{n}}\right\|^2_{\mathbf{H}^{1/2}(\Gamma_{\mathrm{obs}})} + \|p\|^2_{\mathrm{H}^{1/2}(\Gamma_{\mathrm{obs}})} \leq C\left(\|\sigma(\boldsymbol{v}, p)\mathbf{n}\|^2_{\mathbf{H}^{1/2}(\Gamma_{\mathrm{obs}})} + \|\boldsymbol{v}\|^2_{\mathbf{H}^{3/2}(\Gamma_{\mathrm{obs}})} + \|d\|^2_{\mathrm{H}^1(\Omega)} \right). \quad (3.19)$$

Then, (3.12) and (3.13) follow from (3.16), (3.17) and (3.19). ☐

3.3 Proof of the stability estimates for the Navier-Stokes equations

Theorem 1.2 is a simple consequence of Theorem 1.1 applied to the pair $(\boldsymbol{v}\,,\,p) \stackrel{\text{def}}{=} (\boldsymbol{z}_1 - \boldsymbol{z}_2\,,\,\pi_1 - \pi_2)$ which is solution of:

$$\begin{cases} -\nu\Delta\boldsymbol{v} + (\boldsymbol{z}_1\cdot\nabla)\,\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\,\boldsymbol{z}_2 + \nabla p &= \mathbf{0} & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{v} &= 0 & \text{in } \Omega, \\ \boldsymbol{v} &= \boldsymbol{z}_1 - \boldsymbol{z}_2 & \text{on } \Gamma_{\text{obs}}, \\ \sigma(\boldsymbol{v},p)\mathbf{n} &= \sigma(\boldsymbol{z}_1,\pi_1)\mathbf{n} - \sigma(\boldsymbol{z}_2,\pi_2)\mathbf{n} & \text{on } \Gamma_{\text{obs}}. \end{cases}$$

Note that in the same way, we can also obtain the same estimates as in Theorem 3.2 and Theorem 3.5 for a distributed observation.

4 Application: stability estimates for boundary coefficients inverse problems

In the present section, we focus on the proof of Theorem 1.3. We begin by considering the Navier boundary conditions. One can first notice that the pair $(\boldsymbol{v}\,,\,p) \stackrel{\text{def}}{=} (\boldsymbol{z}_1 - \boldsymbol{z}_2\,,\,\pi_1 - \pi_2)$ satisfies

$$\begin{cases} -\nu\Delta\boldsymbol{v} + (\boldsymbol{z}_1\cdot\nabla)\,\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\,\boldsymbol{z}_2 + \nabla p &= \mathbf{0} & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{v} &= 0 & \text{in } \Omega, \\ \boldsymbol{v}\cdot\mathbf{n} &= 0 & \text{on } \Gamma_0, \\ 2\nu\left[\mathcal{D}(\boldsymbol{v})\mathbf{n}\right]_\tau + \alpha_1\boldsymbol{z}_1 - \alpha_2\boldsymbol{z}_2 &= \mathbf{0} & \text{on } \Gamma_0. \end{cases} \tag{4.1}$$

Then, since $(\alpha_2 - \alpha_1)\boldsymbol{z}_1 = \alpha_2\boldsymbol{v} + 2\nu\left[\mathcal{D}(\boldsymbol{v})\mathbf{n}\right]_\tau$ and $(\alpha_2 - \alpha_1)\boldsymbol{z}_2 = \alpha_1\boldsymbol{v} + 2\nu\left[\mathcal{D}(\boldsymbol{v})\mathbf{n}\right]_\tau$ on Γ_0 and since $|\boldsymbol{z}_1| \geq m$ or $|\boldsymbol{z}_2| \geq m$ on $\mathcal{K}$, we have

$$\|\alpha_1 - \alpha_2\|_{\mathrm{L}^2(\mathcal{K})} \leq \frac{C}{m}\left(\|\boldsymbol{v}\|_{\mathbf{L}^2(\Gamma_0)} + \|\nabla\boldsymbol{v}\|_{\mathrm{L}^2(\Gamma_0)}\right). \tag{4.2}$$

To estimate the above right hand side, we use the following inequalities:

$$\|\boldsymbol{v}\|_{\mathbf{L}^2(\Gamma_0)} \leq C\,\|\boldsymbol{v}\|_{\mathbf{L}^2(\Omega)}^{1/2}\,\|\boldsymbol{v}\|_{\mathbf{H}^1(\Omega)}^{1/2} \qquad \text{and} \qquad \|\nabla\boldsymbol{v}\|_{\mathrm{L}^2(\Gamma_0)} \leq C\,\|\boldsymbol{v}\|_{\mathbf{H}^1(\Omega)}^{1/2}\,\|\boldsymbol{v}\|_{\mathbf{H}^2(\Omega)}^{1/2}. \tag{4.3}$$

Note that the above inequalities are an immediate consequence of the interpolation inequality $\|\cdot\|_{\mathrm{L}^2(\partial\Omega)}^2 \leq C\,\|\cdot\|_{\mathrm{H}^1(\Omega)}\,\|\cdot\|_{\mathrm{L}^2(\Omega)}$ which can be obtained for instance by first applying [19, Theorem 1.5.1.10] to get $C > 0$ such that for all $u \in \mathrm{H}^1(\partial\Omega)$ and all $0 < \varepsilon < 1$,

$$\|u\|_{\mathrm{L}^2(\partial\Omega)}^2 \leq C\left(\varepsilon^{1/2}\,\|\nabla u\|_{\mathbf{L}^2(\Omega)}^2 + \varepsilon^{-1/2}\,\|u\|_{\mathrm{L}^2(\Omega)}^2\right),$$

and next by taking $\varepsilon = \|u\|_{\mathrm{L}^2(\Omega)}^2 / \|u\|_{\mathrm{H}^1(\Omega)}^2$. Combining the interpolation inequality $\|\boldsymbol{v}\|_{\mathbf{H}^1(\Omega)} \leq \|\boldsymbol{v}\|_{\mathbf{L}^2(\Omega)}^{1/2}\,\|\boldsymbol{v}\|_{\mathbf{H}^2(\Omega)}^{1/2}$ with the second inequality in (4.3), we deduce that

$$\|\nabla\boldsymbol{v}\|_{\mathrm{L}^2(\Gamma_0)} \leq C\,\|\boldsymbol{v}\|_{\mathbf{L}^2(\Omega)}^{1/4}\,\|\boldsymbol{v}\|_{\mathbf{H}^2(\Omega)}^{3/4}.$$

Hence, from (4.2) we obtain:

$$\|\alpha_1 - \alpha_2\|_{\mathrm{L}^2(\mathcal{K})} \leq \frac{C}{m}\,\|\boldsymbol{v}\|_{\mathbf{L}^2(\Omega)}^{1/4}\,\|\boldsymbol{v}\|_{\mathbf{H}^2(\Omega)}^{3/4} \tag{4.4}$$

and we conclude using the estimate on $\|\boldsymbol{v}\|_{\mathbf{L}^2(\Omega)}$ given by Theorem 1.2.

For the Robin boundary conditions, we proceed in exactly the same way to obtain

$$\|\alpha_1 - \alpha_2\|_{\mathrm{L}^2(\mathcal{K})} \leq \frac{C}{m}\left(\|\boldsymbol{v}\|_{\mathbf{L}^2(\Gamma_0)} + \|\nabla\boldsymbol{v}\|_{\mathrm{L}^2(\Gamma_0)} + \|p\|_{\mathrm{L}^2(\Gamma_0)}\right)$$

and conclude using the estimate on $\|\boldsymbol{v}\|_{\mathbf{L}^2(\Omega)}$ and on $\|p\|_{\mathrm{L}^2(\Omega)}$ given by Theorem 1.2

Remark 4.1. *We can obtain a better estimate assuming more regularity on* $(\boldsymbol{v}, p)$*. For example, if* $(\boldsymbol{v}, p) \in \mathbf{H}^k(\Omega) \times \mathrm{H}^{k-1}(\Omega)$*,* $k \geq 2$*, we can use an interpolation inequality in* (4.4) *to obtain*

$$\|\alpha_1 - \alpha_2\|_{\mathrm{L}^2(\mathcal{K})} \leq \frac{C}{m} \|\boldsymbol{v}\|^{1/4}_{\mathbf{L}^2(\Omega)} \left(\|\boldsymbol{v}\|^{1-2/k}_{\mathbf{L}^2(\Omega)} \|\boldsymbol{v}\|^{2/k}_{\mathbf{H}^k(\Omega)}\right)^{3/4} = \frac{C}{m} \|\boldsymbol{v}\|^{1-3/(2k)}_{\mathbf{L}^2(\Omega)} \|\boldsymbol{v}\|^{3/(2k)}_{\mathbf{H}^k(\Omega)}. \quad (4.5)$$

Then (1.16) *follows from* (4.5) *with the interpolation inequality* $\|\cdot\|_{\mathrm{H}^{\theta n}(\mathcal{K})} \leq C \|\cdot\|^{1-\theta}_{\mathrm{L}^2(\mathcal{K})} \|\cdot\|^{\theta}_{\mathrm{H}^n(\mathcal{K})}$*.*

Remark 4.2. *Concerning the Navier boundary conditions, we can obtain the same result in a different way, by writing* $[\mathcal{D}(\boldsymbol{v})\mathbf{n}]_\tau$ *in terms of* $\operatorname{curl} \boldsymbol{v}$ *on* Γ_0*. Indeed, since* $\boldsymbol{v} \cdot \mathbf{n} = 0$ *on* Γ_0*,* $\nabla(\boldsymbol{v} \cdot \mathbf{n}) = \dfrac{\partial(\boldsymbol{v} \cdot \mathbf{n})}{\partial \mathbf{n}} \mathbf{n}$ *and then,*

$$\begin{aligned}\operatorname{curl} \boldsymbol{v} \times \mathbf{n} = (\nabla \boldsymbol{v} - {}^t\nabla \boldsymbol{v})\mathbf{n} &= \frac{\partial \boldsymbol{v}}{\partial \mathbf{n}} - \nabla(\boldsymbol{v} \cdot \mathbf{n}) + (\nabla \mathbf{n})\boldsymbol{v} \\ &= \frac{\partial(\boldsymbol{v}_\tau + (\boldsymbol{v} \cdot \mathbf{n})\mathbf{n})}{\partial \mathbf{n}} - \frac{\partial(\boldsymbol{v} \cdot \mathbf{n})}{\partial \mathbf{n}}\mathbf{n} + (\nabla \mathbf{n})\boldsymbol{v}_\tau = \frac{\partial \boldsymbol{v}_\tau}{\partial \mathbf{n}} + (\nabla \mathbf{n})\boldsymbol{v}_\tau.\end{aligned}$$

On the other hand, using the same kind of computations, we have

$$2\mathcal{D}(\boldsymbol{v})\mathbf{n} = (\nabla \boldsymbol{v} + {}^t\nabla \boldsymbol{v})\mathbf{n} = \frac{\partial \boldsymbol{v}}{\partial \mathbf{n}} + \nabla(\boldsymbol{v} \cdot \mathbf{n}) - (\nabla \mathbf{n})\boldsymbol{v} = \frac{\partial \boldsymbol{v}_\tau}{\partial \mathbf{n}} + 2\frac{\partial(\boldsymbol{v} \cdot \mathbf{n})}{\partial \mathbf{n}}\mathbf{n} - (\nabla \mathbf{n})\boldsymbol{v}_\tau.$$

Hence, we obtain that

$$\operatorname{curl} \boldsymbol{v} \times \mathbf{n} = [2\mathcal{D}(\boldsymbol{v})\mathbf{n}]_\tau + 2(\nabla \mathbf{n})\boldsymbol{v}_\tau.$$

Thus, in the previous proof, we can write

$$(\alpha_2 - \alpha_1)\boldsymbol{v}_1 = \alpha_2 \boldsymbol{v} + 2\nu \, [\mathcal{D}(\boldsymbol{v})\mathbf{n}]_\tau = \alpha_2 \boldsymbol{v} + \nu \operatorname{curl} \boldsymbol{v} \times \mathbf{n} - 2\nu(\nabla \mathbf{n})\boldsymbol{v}_\tau$$

and conclude using the estimates (1.9) *and* (1.10) *on* $\|\boldsymbol{v}\|_{\mathbf{L}^2(\Omega)}$ *and* $\|\operatorname{curl} \boldsymbol{v}\|_{(\mathrm{L}^2(\Omega))^{2N-3}}$ *given by Theorem 1.2.*

5 Application to error estimates

In this section, we consider the reconstruction of $(\boldsymbol{v}, p)$, solution of the Stokes system in Ω, knowing $\boldsymbol{v}$ and $\sigma(\boldsymbol{v}, p)\mathbf{n}$ on Γ_{obs}. In other words, we consider the data completion problem for the Stokes system, that is: from given data $\boldsymbol{g}_D \in \mathbf{H}^{3/2}(\Gamma_{\text{obs}})$ and $\boldsymbol{g}_N \in \mathbf{H}^{1/2}(\Gamma_{\text{obs}})$, reconstruct $(\boldsymbol{v}, p) \in \mathbf{H}^2(\Omega) \times \mathrm{H}^1(\Omega)$ verifying

$$\left\{\begin{array}{rcll} -\nu\Delta \boldsymbol{v} + \nabla p &=& \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{v} &=& 0 & \text{in } \Omega, \\ \boldsymbol{v} &=& \boldsymbol{g}_D & \text{on } \Gamma_{\text{obs}}, \\ \sigma(\boldsymbol{v}, p)\mathbf{n} &=& \boldsymbol{g}_N & \text{on } \Gamma_{\text{obs}}. \end{array}\right. \quad (5.1)$$

As the problem is ill-posed, it is mandatory to use a stabilization method to stably reconstruct $(\boldsymbol{v}, p)$ from the data $\boldsymbol{f}$, $\boldsymbol{g}_D$ and $\boldsymbol{g}_N$.

Such a stabilization method usually depends on a parameter of regularization $\varepsilon > 0$, and it must fulfill the two following requirements: it must have a solution for *any data* $\boldsymbol{f}$, $\boldsymbol{g}_D$ and $\boldsymbol{g}_N$, regardless of the existence of a solution to the corresponding Stokes problem (5.1). And its solution should converge to the solution of (5.1) when the parameter ε goes to zero, when such a solution exists.

We study below two standard methods of regularization: a *quasi-reversibility method* and a *penalized Kohn-Vogelius method.* In particular, we obtained the convergence rates of these methods directly from the estimates obtained previously.

In the following, we denote $\Big((\boldsymbol{v},p),(\boldsymbol{w},q)\Big)_{\mathbf{H}^2(\Omega)\times \mathrm{H}^1(\Omega)} \overset{\text{def}}{=} (\boldsymbol{v},\boldsymbol{w})_{\mathbf{H}^2(\Omega)}+(p,q)_{\mathrm{H}^1(\Omega)}$ which is obviously a scalar product on the Hilbert space $\mathbf{H}^2(\Omega)\times \mathrm{H}^1(\Omega)$, and $\|(\boldsymbol{v},p)\|_{\mathbf{H}^2(\Omega)\times \mathrm{H}^1(\Omega)}$ the corresponding norm.

5.1 Error estimates for the quasi-reversibility method

The quasi-reversibility method has been introduced in [23] by Lattès *et al.* to stabilize elliptic, parabolic and hyperbolic ill-posed problem. The main idea of the method is to solve well-posed variational fourth-order problem, depending on ε.

More precisely, for $\varepsilon > 0$, we define the following quasi-reversibility variational problem: find $(\boldsymbol{v}_\varepsilon, p_\varepsilon) \in \mathbf{H}^2(\Omega)\times \mathrm{H}^1(\Omega)$ such that $\boldsymbol{v}_\varepsilon = \boldsymbol{g}_D$ on Γ_{obs}, $\sigma(\boldsymbol{v}_\varepsilon, p_\varepsilon)\mathbf{n} = \boldsymbol{g}_N$ on Γ_{obs} and for all $(\boldsymbol{w},q) \in \mathbf{H}^2(\Omega)\times \mathrm{H}^1(\Omega)$ with $\boldsymbol{w} = \mathbf{0}$ and $\sigma(\boldsymbol{w},q)\mathbf{n} = \mathbf{0}$ on Γ_{obs}, we have:

$$\int_\Omega (-\nu\Delta\boldsymbol{v}_\varepsilon + \nabla p_\varepsilon)\cdot(-\nu\Delta\boldsymbol{w}+\nabla q)\,\mathrm{d}x + \Big(\mathrm{div}(\boldsymbol{v}_\varepsilon), \mathrm{div}(\boldsymbol{w})\Big)_{\mathrm{H}^1(\Omega)} \\ + \varepsilon(\boldsymbol{v}_\varepsilon,\boldsymbol{w})_{\mathbf{H}^2(\Omega)} + \varepsilon(p_\varepsilon,q)_{\mathrm{H}^1(\Omega)} = \int_\Omega \boldsymbol{f}\cdot(-\nu\Delta\boldsymbol{w}+\nabla q)\,\mathrm{d}x. \quad (5.2)$$

We start by proving that the quasi-reversibility problem is well-posed:

Proposition 5.1. *For any* $(\boldsymbol{f},\boldsymbol{g}_D,\boldsymbol{g}_N) \in \mathbf{L}^2(\Omega)\times\mathbf{H}^{3/2}(\Gamma_{\text{obs}})\times\mathbf{H}^{1/2}(\Gamma_{\text{obs}})$*, there exists a unique solution* $(\boldsymbol{v}_\varepsilon,p_\varepsilon)\in\mathbf{H}^2(\Omega)\times\mathrm{H}^1(\Omega)$ *to the quasi-reversibility problem* (5.2).

Proof. Let us first note that there exists $(\boldsymbol{V},P)\in\mathbf{H}^2(\Omega)\times\mathrm{H}^1(\Omega)$ such that $\boldsymbol{V}_{|\Gamma_{\text{obs}}} = \boldsymbol{g}_D$, $\sigma(\boldsymbol{V},P)\mathbf{n}_{|\Gamma_{\text{obs}}} = \boldsymbol{g}_N$ and

$$\|(\boldsymbol{V},P)\|_{\mathbf{H}^2(\Omega)\times\mathrm{H}^1(\Omega)} \leq C\|(\boldsymbol{g}_D,\boldsymbol{g}_N)\|_{\mathbf{H}^{3/2}(\Gamma_{\text{obs}})\times\mathbf{H}^{1/2}(\Gamma_{\text{obs}})}.$$

Indeed, since $\sigma(\boldsymbol{V},P))\mathbf{n}\cdot\mathbf{n} = 2\nu\frac{\partial\boldsymbol{V}}{\partial\mathbf{n}}\cdot\mathbf{n} - P$ and

$$[\sigma(\boldsymbol{V},P)\mathbf{n}]_\tau = \nu\left(\frac{\partial\boldsymbol{V}_\tau}{\partial\mathbf{n}} + \nabla_\tau(\boldsymbol{V}\cdot\mathbf{n}) - (\nabla\mathbf{n})\boldsymbol{V}_\tau\right) = \nu\left(\frac{\partial\boldsymbol{V}_\tau}{\partial\mathbf{n}} + (\nabla_\tau\boldsymbol{V})\mathbf{n}\right),$$

it suffices to choose $P=0$ and a continuous lifting $\boldsymbol{V}$ which satisfies $\boldsymbol{V}=\boldsymbol{g}_D$ and $\nu\frac{\partial\boldsymbol{V}}{\partial\mathbf{n}} = \frac{1}{2}(\boldsymbol{g}_N\cdot\mathbf{n})\mathbf{n} + \boldsymbol{g}_{N\tau} - \nu(\nabla_\tau\boldsymbol{g}_D)\mathbf{n}$ on Γ_{obs}.

Defining $(\widetilde{\boldsymbol{v}}_\varepsilon\,,\,\widetilde{p}_\varepsilon) \overset{\text{def}}{=} (\boldsymbol{v}_\varepsilon - \boldsymbol{V}\,,\,p_\varepsilon - P)$, we see that $\widetilde{\boldsymbol{v}}_\varepsilon = \mathbf{0}$ and $\sigma(\widetilde{\boldsymbol{v}_\varepsilon},\widetilde{p_\varepsilon})\mathbf{n} = \mathbf{0}$ on Γ_{obs} and, for all $(\boldsymbol{w},q)\in\mathbf{H}^2(\Omega)\times\mathrm{H}^1(\Omega)$ such that $\boldsymbol{w}=\mathbf{0}$ and $\sigma(\boldsymbol{w},q)\mathbf{n}=\mathbf{0}$ on Γ_{obs}, we have

$$\int_\Omega (-\nu\Delta\widetilde{\boldsymbol{v}}_\varepsilon + \nabla\widetilde{p}_\varepsilon)\cdot(-\nu\Delta\boldsymbol{w}+\nabla q)\,\mathrm{d}x + \Big(\mathrm{div}(\widetilde{\boldsymbol{v}}_\varepsilon),\mathrm{div}(\boldsymbol{w})\Big)_{\mathrm{H}^1(\Omega)} \\ + \varepsilon(\widetilde{\boldsymbol{v}}_\varepsilon,\boldsymbol{w})_{\mathbf{H}^2(\Omega)} + \varepsilon(\widetilde{p}_\varepsilon,q)_{\mathrm{H}^1(\Omega)} = \int_\Omega \widetilde{\boldsymbol{f}}\cdot(-\nu\Delta\boldsymbol{w}+\nabla q)\,\mathrm{d}x - \int_\Omega \mathrm{div}(\boldsymbol{V})\,\mathrm{div}(\boldsymbol{w})\,\mathrm{d}x \\ - \varepsilon(\boldsymbol{V},\boldsymbol{w})_{\mathbf{H}^2(\Omega)} - \varepsilon(P,q)_{\mathrm{H}^1(\Omega)},$$

where $\widetilde{\boldsymbol{f}} \overset{\text{def}}{=} \boldsymbol{f} + \nu\Delta\boldsymbol{V} - \nabla P$. The Lax-Milgram theorem gives then the result. □

Suppose now that the initial data completion problem admits a (necessarily unique) solution $(\boldsymbol{v},p)\in\mathbf{H}^2(\Omega)\times\mathrm{H}^1(\Omega)$. Then, we have the following

Theorem 5.2. *The solution* $(\boldsymbol{v}_\varepsilon, p_\varepsilon) \in \mathbf{H}^2(\Omega) \times \mathrm{H}^1(\Omega)$ *of the quasi-reversibility problem* (5.2) *converges to* $(\boldsymbol{v}, p) \in \mathbf{H}^2(\Omega) \times \mathrm{H}^1(\Omega)$ *solution of the data completion problem for the Stokes problem* (5.1) *when* ε *tends to zero, strongly in* $\mathbf{H}^2(\Omega) \times \mathrm{H}^1(\Omega)$. *We furthermore have the estimate*

$$\| -\nu \Delta \boldsymbol{v}_\varepsilon + \nabla p_\varepsilon - \boldsymbol{f} \|^2_{\mathbf{L}^2(\Omega)} + \| \mathrm{div}(\boldsymbol{v}_\varepsilon) \|^2_{\mathrm{H}^1(\Omega)} \leq \varepsilon \| (\boldsymbol{v}, p) \|^2_{\mathbf{H}^2(\Omega) \times \mathrm{H}^1(\Omega)}. \tag{5.3}$$

Proof. Using $(\boldsymbol{w}, q) \overset{\mathrm{def}}{=} (\boldsymbol{v}_\varepsilon - \boldsymbol{v}\, p_\varepsilon - p)$ as test functions in the quasi-reversibility problem (5.2), which is admissible as they verify the limit conditions, we directly obtain

$$\| -\nu \Delta \boldsymbol{v}_\varepsilon + \nabla p_\varepsilon - \boldsymbol{f} \|^2_{\mathbf{L}^2(\Omega)} + \| \mathrm{div}(\boldsymbol{v}_\varepsilon) \|^2_{\mathrm{H}^1(\Omega)} + \varepsilon \Big((\boldsymbol{v}_\varepsilon, p_\varepsilon), (\boldsymbol{v}_\varepsilon - \boldsymbol{v}, p_\varepsilon - p) \Big)_{\mathbf{H}^2(\Omega) \times \mathrm{H}^1(\Omega)} = 0. \tag{5.4}$$

We hence have $\Big((\boldsymbol{v}_\varepsilon, p_\varepsilon), (\boldsymbol{v}_\varepsilon - \boldsymbol{v}, p_\varepsilon - p) \Big)_{\mathbf{H}^2(\Omega) \times \mathrm{H}^1(\Omega)} \leq 0$ which implies

$$\| (\boldsymbol{v}_\varepsilon, p_\varepsilon) \|_{\mathbf{H}^2(\Omega) \times \mathrm{H}^1(\Omega)} \leq \| (\boldsymbol{v}, p) \|_{\mathbf{H}^2(\Omega) \times \mathrm{H}^1(\Omega)}. \tag{5.5}$$

Subtracting $\varepsilon \Big((\boldsymbol{v}, p), (\boldsymbol{v}_\varepsilon - \boldsymbol{v}, p_\varepsilon - p) \Big)_{\mathbf{H}^2(\Omega) \times \mathrm{H}^1(\Omega)}$ to equation (5.4), we obtain

$$\| (\boldsymbol{v}_\varepsilon - \boldsymbol{v}, p_\varepsilon - p) \|^2_{\mathbf{H}^2(\Omega) \times \mathrm{H}^1(\Omega)} \leq - \Big((\boldsymbol{v}, p), (\boldsymbol{v}_\varepsilon - \boldsymbol{v}, p_\varepsilon - p) \Big)_{\mathbf{H}^2(\Omega) \times \mathrm{H}^1(\Omega)} \tag{5.6}$$

implying

$$\| (\boldsymbol{v}_\varepsilon - \boldsymbol{v}, p_\varepsilon - p) \|_{\mathbf{H}^2(\Omega) \times \mathrm{H}^1(\Omega)} \leq \| (\boldsymbol{v}, p) \|_{\mathbf{H}^2(\Omega) \times \mathrm{H}^1(\Omega)}. \tag{5.7}$$

Going back to equation (5.4), we finally obtain

$$\| -\nu \Delta \boldsymbol{v}_\varepsilon + \nabla p_\varepsilon - \boldsymbol{f} \|^2_{\mathbf{L}^2(\Omega)} + \| \mathrm{div}(\boldsymbol{v}_\varepsilon) \|^2_{\mathrm{H}^1(\Omega)} \leq \varepsilon \left| \Big((\boldsymbol{v}_\varepsilon, p_\varepsilon), (\boldsymbol{v}_\varepsilon - \boldsymbol{v}, p_\varepsilon - p) \Big)_{\mathbf{H}^2(\Omega) \times \mathrm{H}^1(\Omega)} \right|$$

which, using (5.5) and (5.7), directly leads to the estimate (5.3).

Now, suppose that $\boldsymbol{v}_\varepsilon$ and p_ε do not converge to $\boldsymbol{v}$ and p. Then there exist $\rho > 0$ and ε_n, sequence of strictly positive real numbers verifying $\varepsilon_n \xrightarrow[n \to \infty]{} 0$, such that the couple $(\boldsymbol{v}_n \overset{\mathrm{def}}{=} \boldsymbol{v}_{\varepsilon_n}, p_n \overset{\mathrm{def}}{=} p_{\varepsilon_n})$ satisfies

$$\| \boldsymbol{v}_n - \boldsymbol{v}, p_n - p \|_{\mathbf{H}^2(\Omega) \times \mathrm{H}^1(\Omega)} > \rho.$$

By equation (5.5), we know that $(\boldsymbol{v}_n, p_n)$ is a bounded sequence in $\mathbf{H}^2(\Omega) \times \mathrm{H}^1(\Omega)$. Hence, up to a subsequence (that we still denote $(\boldsymbol{v}_n, p_n)$) the sequence converges to some $(\boldsymbol{w}, q)$ weakly in $\mathbf{H}^2(\Omega) \times \mathrm{H}^1(\Omega)$. Then equation (5.3) and the limit conditions verified by $(\boldsymbol{v}_n, p_n)$ directly imply that $(\boldsymbol{w}, q)$ verifies the Stokes data completion problem (5.1), which in turn implies by uniqueness of such solution that $\boldsymbol{w} = \boldsymbol{v}$ and $q = p$. Therefore, $\boldsymbol{v}_n$ weakly converges to $\boldsymbol{v}$ in $\mathbf{H}^2(\Omega)$ and p_n weakly converges to p in $\mathrm{H}^1(\Omega)$. But Equation (5.5) implies then that $(\boldsymbol{v}_n, p_n)$ strongly converges to $(\boldsymbol{v}, p)$, which is a direct contradiction with the definition of the sequence, and therefore ends the proof. □

Remark 5.3. *It is not difficult to obtain the following complement to the theorem: if the initial data completion problem for the Stokes system does not admit a solution, then*

$$\| (\boldsymbol{v}_\varepsilon, p_\varepsilon) \|_{\mathbf{H}^2(\Omega) \times \mathrm{H}^1(\Omega)} \xrightarrow[\varepsilon \to 0]{} +\infty.$$

Otherwise, we would have a sequence of strictly positive real numbers $(\varepsilon_n)_{n\in\mathbb{N}}$ *verifying* $\varepsilon_n \xrightarrow[n\to\infty]{} 0$ *and* $\|(\boldsymbol{v}_{\varepsilon_n}, p_{\varepsilon_n})\|_{\mathbf{H}^2(\Omega)\times\mathrm{H}^1(\Omega)} \leq C$. *But using the same arguments as in the last paragraph of the proof of theorem 5.2, extracting a subsequence and passing to the limit, we would obtain a solution to the data completion problem for the Stokes system, in obvious contradiction with the assumption.*

Proposition 5.1 and Theorem 5.2 clearly show that the proposed quasi-reversibility method (5.2) is a regularization method for problem (5.1). However, if Theorem 5.2 assures the convergence of the approximated solution to the exact one, it does not give any rate of convergence. Actually, it is known (see [22, section 2.5] and the references therein) that Carleman estimates are the key argument to derive convergence rates for the quasi-reversibility method. This is the case for the quasi-reversibility method proposed above and we now prove Theorem 1.6 for this method:

Proof of Theorem 1.6 for the quasi-reversibility method. Defining $(\boldsymbol{u}\,,\,q) \overset{\text{def}}{=} (\boldsymbol{v}_\varepsilon - \boldsymbol{v}\,,\,p_\varepsilon - p)$, we notice that we have $\boldsymbol{u} = \mathbf{0}$ and $\sigma(\boldsymbol{u}, q)\mathbf{n} = \mathbf{0}$ on Γ_{obs} and that the following estimates hold (see Inequalities (5.3) and (5.7)):

$$\begin{aligned}
\|(\boldsymbol{u}, q)\|_{\mathbf{H}^2(\Omega)\times\mathrm{H}^1(\Omega)} &\leq \|(\boldsymbol{v}, p)\|_{\mathbf{H}^2(\Omega)\times\mathrm{H}^1(\Omega)} \\
\|-\nu\Delta\boldsymbol{u} + \nabla q\|_{\mathbf{L}^2(\Omega)} &\leq \sqrt{\varepsilon}\|(\boldsymbol{v}, p)\|_{\mathbf{H}^2(\Omega)\times\mathrm{H}^1(\Omega)} \\
\|\mathrm{div}(\boldsymbol{u})\|_{\mathrm{L}^2(\Omega)} &\leq \sqrt{\varepsilon}\|(\boldsymbol{v}, p)\|_{\mathbf{H}^2(\Omega)\times\mathrm{H}^1(\Omega)}.
\end{aligned}$$

Hence, applying estimates (1.5) and (1.6), we directly obtain the result. □

Remark 5.4. *Suppose that instead of exact data* $(\boldsymbol{f}, \boldsymbol{g}_D, \boldsymbol{g}_N) \in \mathbf{L}^2(\Omega_{\text{obs}}) \times \mathbf{H}^{3/2}(\Gamma_{\text{obs}}) \times \mathbf{H}^{1/2}(\Gamma_{\text{obs}})$, *with corresponding solution* $(\boldsymbol{v}, p) \in \mathbf{H}^2(\Omega) \times \mathrm{H}^1(\Omega)$, *we have noisy data* $(\boldsymbol{f}^\delta, \boldsymbol{g}_D^\delta, \boldsymbol{g}_n^\delta) \in \mathbf{L}^2(\Omega) \times \mathbf{H}^{3/2}(\Gamma_{\text{obs}}) \times \mathbf{H}^{1/2}(\Gamma_{\text{obs}})$, *such that*

$$\|\boldsymbol{f}^\delta - \boldsymbol{f}\|_{\mathbf{L}^2(\Omega)} \leq \delta, \quad \|\boldsymbol{g}_D^\delta - \boldsymbol{g}_D\|_{\mathbf{H}^{3/2}(\Gamma_{\text{obs}})} \leq \delta \quad \textit{and} \quad \|\boldsymbol{g}_N^\delta - \boldsymbol{g}_N\|_{\mathbf{H}^{1/2}(\Gamma_{\text{obs}})} \leq \delta.$$

Due to the ill-posedness of the data completion problem (5.1), *there might be no solution corresponding to this noisy data. However, the quasi-reversibility problem* (5.2) *has a corresponding solution, denoted* $\boldsymbol{v}_\varepsilon^\delta$ *and* p_ε^δ. *We also denote* $\boldsymbol{v}_\varepsilon$ *and* p_ε *the solution of the quasi-reversibility problem with exact data. It is not difficult to verify that there exists a constant* $C > 0$, *depending only on the geometry of the domain, such that*

$$\|(\boldsymbol{v}_\varepsilon^\delta - \boldsymbol{v}_\varepsilon, p_\varepsilon^\delta - p_\varepsilon)\|_{\mathbf{H}^2(\Omega)\times\mathrm{H}^1(\Omega)} \leq C\frac{\delta}{\sqrt{\varepsilon}}.$$

Combining this result with the previous estimates, we therefore obtain

$$\|(\boldsymbol{v}_\varepsilon^\delta - \boldsymbol{v}, p_\varepsilon^\delta - p)\|_{\mathbf{H}^1(\Omega)\times\mathrm{L}^2(\Omega)} \leq C\frac{\delta}{\sqrt{\varepsilon}} + \frac{C(\boldsymbol{v}, p)}{\left(\ln(1 + \frac{\widetilde{C}(\boldsymbol{v},p)}{\sqrt{\varepsilon}})\right)^{1/2}}.$$

Such estimate highlight the competition between regularization and noise, which leads to the question of the optimal choice of the regularization parameter ε *with respect to the amplitude of the noise* δ. *On this subject of the optimal choice of the regularization parameter for the quasi-reversibility method for elliptic equations, see [10, 11] and the references therein.*

5.2 Error estimates for the Kohn-Vogelius method

The quasi-reversibility method proposed in the previous section regularizes the data completion problem for the Stokes system by solving approximately the first two equations of (5.1) (see the estimate in Theorem 5.2) while verifying exactly the boundary conditions. The Kohn-Vogelius method we study now is somehow a symmetric method, in the sense that it solves exactly the equations in Ω with approximated boundary conditions. And again, we obtain the rate of convergence of the method using the same estimates (1.5) and (1.6).

We recall that $\Gamma_{\text{obs}}^C \overset{\text{def}}{=} \partial\Omega\backslash\overline{\Gamma_{\text{obs}}}$. For $\boldsymbol{\varphi}_N \in \mathbf{H}^{1/2}(\Gamma_{\text{obs}})$ and $\boldsymbol{\psi}_D \in \mathbf{H}^{3/2}(\Gamma_{\text{obs}})$, we denote $(\boldsymbol{v}_{\boldsymbol{\varphi}_N}, p_{\boldsymbol{\varphi}_N}) \in \mathbf{H}^2(\Omega) \times \mathrm{H}^1(\Omega)$ and $(\boldsymbol{v}_{\boldsymbol{\psi}_D}, p_{\boldsymbol{\psi}_D}) \in \mathbf{H}^2(\Omega) \times \mathrm{H}^1(\Omega)$ the respective solutions of

$$\left\{\begin{array}{rcll} -\nu\Delta \boldsymbol{v}_{\boldsymbol{\varphi}_N} + \nabla p_{\boldsymbol{\varphi}_N} & = & \boldsymbol{f} & \text{in } \Omega, \\ \text{div } \boldsymbol{v}_{\boldsymbol{\varphi}_N} & = & 0 & \text{in } \Omega, \\ \boldsymbol{v}_{\boldsymbol{\varphi}_N} & = & \boldsymbol{g}_D & \text{on } \Gamma_{\text{obs}}, \\ \sigma(\boldsymbol{v}_{\boldsymbol{\varphi}_N}, p_{\boldsymbol{\varphi}_N})\mathbf{n} & = & \boldsymbol{\varphi}_N & \text{on } \Gamma_{\text{obs}}^C, \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{rcll} -\nu\Delta \boldsymbol{v}_{\boldsymbol{\psi}_D} + \nabla p_{\boldsymbol{\psi}_D} & = & \boldsymbol{f} & \text{in } \Omega, \\ \text{div } \boldsymbol{v}_{\boldsymbol{\psi}_D} & = & 0 & \text{in } \Omega, \\ \sigma(\boldsymbol{v}_{\boldsymbol{\psi}_D}, p_{\boldsymbol{\psi}_D})\mathbf{n} & = & \boldsymbol{g}_N & \text{on } \Gamma_{\text{obs}}, \\ \boldsymbol{v}_{\boldsymbol{\psi}_D} & = & \boldsymbol{\psi}_D & \text{on } \Gamma_{\text{obs}}^C. \end{array}\right. \tag{5.8}$$

We define the non-negative functional

$$F\,:\,(\boldsymbol{\varphi}_N, \boldsymbol{\psi}_D) \in \mathbf{H}^{1/2}(\Gamma_{\text{obs}}^C) \times \mathbf{H}^{3/2}(\Gamma_{\text{obs}}^C) \mapsto |\boldsymbol{v}_{\boldsymbol{\varphi}_N} - \boldsymbol{v}_{\boldsymbol{\psi}_D}|^2_{\mathbf{H}^2(\Omega)} + |\boldsymbol{v}_{\boldsymbol{\varphi}_N} - \boldsymbol{v}_{\boldsymbol{\psi}_D}|^2_{\mathbf{H}^1(\Omega)} \in \mathbb{R},$$

where $|.|_{\mathbf{H}^i(\Omega)}$ $(i = 1, 2)$ is the $\mathbf{H}^i$-seminorm. It is not difficult to verify that the two following propositions are equivalent:

- there exists $(\boldsymbol{\varphi}_N, \boldsymbol{\psi}_D) \in \mathbf{H}^{1/2}(\Gamma_{\text{obs}}^C) \times \mathbf{H}^{3/2}(\Gamma_{\text{obs}}^C)$ such that $F(\boldsymbol{\varphi}_N, \boldsymbol{\psi}_D) = 0$;
- there exists a (necessarily unique) solution to the data completion problem (5.1).

Hence one could try to reconstruct the solution of problem (5.1) by minimizing F. However, this is not a stable strategy: indeed, due to the denseness of the non-admissible data (*i.e.* the data for which the problem (5.1) has no solution) in $\mathbf{L}^2(\Omega) \times \mathbf{H}^{3/2}(\Gamma_{\text{obs}}) \times \mathbf{H}^{1/2}(\Gamma_{\text{obs}})$ (see [7] or [15, section 2]), the minimum of F is always 0, but there are minimizing sequences $(\boldsymbol{\varphi}_N^m, \boldsymbol{\psi}_D^m)$ such that

$$\lim_{m\to\infty} \|(\boldsymbol{\varphi}_N^m, \boldsymbol{\psi}_D^m)\|_{\mathbf{H}^{1/2}(\Gamma_{\text{obs}}^C)\times\mathbf{H}^{3/2}(\Gamma_{\text{obs}}^C)} = +\infty.$$

Therefore, to regularize the problem, we add a penalization term: for $\varepsilon > 0$, we introduce the functional $F_\varepsilon\,:\,\mathbf{H}^{1/2}(\Gamma_{\text{obs}}^C) \times \mathbf{H}^{3/2}(\Gamma_{\text{obs}}^C) \to \mathbb{R}$ defined by

$$F_\varepsilon(\boldsymbol{\varphi}_N, \boldsymbol{\psi}_D) = F(\boldsymbol{\varphi}_N, \boldsymbol{\psi}_D) + \varepsilon\|(\boldsymbol{v}_{\boldsymbol{\varphi}_N}, p_{\boldsymbol{\varphi}_N})\|^2_{\mathbf{H}^2(\Omega)\times\mathrm{H}^1(\Omega)} + \varepsilon\|(\boldsymbol{v}_{\boldsymbol{\psi}_D}, p_{\boldsymbol{\psi}_D})\|^2_{\mathbf{H}^2(\Omega)\times\mathrm{H}^1(\Omega)}.$$

We have the following result:

Proposition 5.5. *For any* $(\boldsymbol{f}, \boldsymbol{g}_D, \boldsymbol{g}_N) \in \mathbf{L}^2(\Omega) \times \mathbf{H}^{3/2}(\Gamma_{\text{obs}}) \times \mathbf{H}^{1/2}(\Gamma_{\text{obs}})$*, there exists a unique* $(\boldsymbol{\varphi}_N^\varepsilon, \boldsymbol{\psi}_D^\varepsilon) \in \mathbf{H}^{1/2}(\Gamma_{\text{obs}}^C) \times \mathbf{H}^{3/2}(\Gamma_{\text{obs}}^C)$ *such that*

$$F_\varepsilon(\boldsymbol{\varphi}_N^\varepsilon, \boldsymbol{\psi}_D^\varepsilon) = \min_{(\boldsymbol{\varphi}_N, \boldsymbol{\psi}_D)\in\mathbf{H}^{1/2}(\Gamma_{\text{obs}}^C)\times\mathbf{H}^{3/2}(\Gamma_{\text{obs}}^C)} F_\varepsilon(\boldsymbol{\varphi}_N, \boldsymbol{\psi}_D).$$

Proof. Obviously, the functional F_ε is continuous and strictly convex. Furthermore, it is coercive. Indeed, suppose it is not. Then there exists a sequence $(\boldsymbol{\varphi}_N^m, \boldsymbol{\psi}_D^m)$ and a constant $C > 0$ such that

$$\lim_{m\to\infty} \|(\boldsymbol{\varphi}_N^m, \boldsymbol{\psi}_D^m)\|_{\mathbf{H}^{1/2}(\Gamma_{\text{obs}}^C)\times\mathbf{H}^{3/2}(\Gamma_{\text{obs}}^C)} = +\infty \quad \text{and} \quad F_\varepsilon(\boldsymbol{\varphi}_N^m, \boldsymbol{\psi}_D^m) < C.$$

This directly implies $\|(\boldsymbol{v}_{\boldsymbol{\varphi}_N^m}, p_{\boldsymbol{\varphi}_N^m})\|_{\mathbf{H}^2(\Omega)\times \mathrm{H}^1(\Omega)} < C$ and $\|(\boldsymbol{v}_{\boldsymbol{\psi}_D^m}, p_{\boldsymbol{\psi}_D^m})\|_{\mathbf{H}^2(\Omega)\times \mathrm{H}^1(\Omega)} < C$, which directly implies $\|(\boldsymbol{\varphi}_N^m, \boldsymbol{\psi}_D^m)\|_{\mathbf{H}^{1/2}(\Gamma_{\text{obs}}^C)\times \mathbf{H}^{3/2}(\Gamma_{\text{obs}}^C)} < C$ by continuity of trace and normal derivative operators, which is a contradiction with the initial assumptions.

Therefore F_ε is continuous, strictly convex and coercive, which implies the result (see [16]). ☐

Suppose now that the initial data completion problem admits a (necessarily unique) solution $(\boldsymbol{v}, p) \in \mathbf{H}^2(\Omega) \times \mathrm{H}^1(\Omega)$. Then, we have the following

Theorem 5.6. *The solution* $(\boldsymbol{v}_{\boldsymbol{\varphi}_N^\varepsilon}, p_{\boldsymbol{\psi}_D^\varepsilon}) \in \mathbf{H}^2(\Omega) \times \mathrm{H}^1(\Omega)$ *converges to* $(\boldsymbol{v}, p) \in \mathbf{H}^2(\Omega) \times \mathrm{H}^1(\Omega)$ *solution of the data completion problem for the Stokes problem* (5.1) *when* ε *tends to zero, strongly in* $\mathbf{H}^2(\Omega) \times \mathrm{H}^1(\Omega)$.

Proof. We denote $\boldsymbol{\varphi}_N^{\text{ex}} \overset{\text{def}}{=} \sigma(\boldsymbol{v}, p)\mathbf{n}_{|\Gamma_{\text{obs}}^C}$ and $\boldsymbol{\psi}_D^{\text{ex}} \overset{\text{def}}{=} \boldsymbol{v}_{|\Gamma_{\text{obs}}^C}$. By definition, we have $(\boldsymbol{v}_{\boldsymbol{\varphi}_N^{\text{ex}}}, p_{\boldsymbol{\varphi}_N^{\text{ex}}}) = (\boldsymbol{v}_{\boldsymbol{\psi}_D^{\text{ex}}}, p_{\boldsymbol{\psi}_D^{\text{ex}}}) = (\boldsymbol{v}, p)$ and $F(\boldsymbol{\varphi}_N^{\text{ex}}, \boldsymbol{\psi}_D^{\text{ex}}) = 0$. Therefore, by definition of $\boldsymbol{\varphi}_N^\varepsilon$ and $\boldsymbol{\psi}_D^\varepsilon$, we have

$$|\boldsymbol{v}_{\boldsymbol{\varphi}_N^\varepsilon} - \boldsymbol{v}_{\boldsymbol{\psi}_D^\varepsilon}|^2_{\mathbf{H}^2(\Omega)} + |\boldsymbol{v}_{\boldsymbol{\varphi}_N^\varepsilon} - \boldsymbol{v}_{\boldsymbol{\psi}_D^\varepsilon}|^2_{\mathbf{H}^1(\Omega)} + \varepsilon\|(\boldsymbol{v}_{\boldsymbol{\varphi}_N^\varepsilon}, p_{\boldsymbol{\varphi}_N^\varepsilon})\|^2_{\mathbf{H}^2(\Omega)\times\mathrm{H}^1(\Omega)} + \varepsilon\|(\boldsymbol{v}_{\boldsymbol{\psi}_D^\varepsilon}, p_{\boldsymbol{\psi}_D^\varepsilon})\|^2_{\mathbf{H}^2(\Omega)\times\mathrm{H}^1(\Omega)}$$
$$\leq F_\varepsilon(\boldsymbol{\varphi}_N^{\text{ex}}, \boldsymbol{\psi}_D^{\text{ex}}) = 2\varepsilon\|(\boldsymbol{v}, p)\|^2_{\mathbf{H}^2(\Omega)\times\mathrm{H}^1(\Omega)}$$

which directly implies

$$|\boldsymbol{v}_{\boldsymbol{\varphi}_N^\varepsilon} - \boldsymbol{v}_{\boldsymbol{\psi}_D^\varepsilon}|^2_{\mathbf{H}^2(\Omega)} + |\boldsymbol{v}_{\boldsymbol{\varphi}_N^\varepsilon} - \boldsymbol{v}_{\boldsymbol{\psi}_D^\varepsilon}|^2_{\mathbf{H}^1(\Omega)} \leq 2\varepsilon\|(\boldsymbol{v}, p)\|^2_{\mathbf{H}^2(\Omega)\times\mathrm{H}^1(\Omega)} \tag{5.9}$$

and

$$\|(\boldsymbol{v}_{\boldsymbol{\varphi}_N^\varepsilon}, p_{\boldsymbol{\varphi}_N^\varepsilon})\|^2_{\mathbf{H}^2(\Omega)\times\mathrm{H}^1(\Omega)} + \|(\boldsymbol{v}_{\boldsymbol{\psi}_D^\varepsilon}, p_{\boldsymbol{\psi}_D^\varepsilon})\|^2_{\mathbf{H}^2(\Omega)\times\mathrm{H}^1(\Omega)} \leq 2\|(\boldsymbol{v}, p)\|^2_{\mathbf{H}^2(\Omega)\times\mathrm{H}^1(\Omega)}. \tag{5.10}$$

Let us consider now an arbitrary sequence of positive real numbers ε_m such that $\lim\limits_{m\to\infty} \varepsilon_m = 0$. From (5.10), we see that

$$(\boldsymbol{v}_{\boldsymbol{\varphi}_N^m}, p_{\boldsymbol{\varphi}_N^m}) \overset{\text{def}}{=} (\boldsymbol{v}_{\boldsymbol{\varphi}_{N\varepsilon_m}}, p_{\boldsymbol{\varphi}_{N\varepsilon_m}}) \quad \text{and} \quad (\boldsymbol{v}_{\boldsymbol{\psi}_D^m}, p_{\boldsymbol{\psi}_D^m}) \overset{\text{def}}{=} (\boldsymbol{v}_{\boldsymbol{\psi}_{D\varepsilon_m}}, p_{\boldsymbol{\psi}_{D\varepsilon_m}})$$

are bounded in $\mathbf{H}^2(\Omega) \times \mathrm{H}^1(\Omega)$. Therefore, up to a subsequence, we have the following weak convergences in $\mathbf{H}^2(\Omega)$

$$\boldsymbol{v}_{\boldsymbol{\psi}_D^m} \rightharpoonup \boldsymbol{v}_{\boldsymbol{\psi}_{D_\infty}}, \quad \boldsymbol{v}_{\boldsymbol{\varphi}_N^m} \rightharpoonup \boldsymbol{v}_{\boldsymbol{\varphi}_{N_\infty}}$$

and the following weak convergences in $\mathrm{H}^1(\Omega)$

$$p_{\boldsymbol{\psi}_D^m} \rightharpoonup p_{\boldsymbol{\psi}_{D_\infty}}, \quad p_{\boldsymbol{\varphi}_N^m} \rightharpoonup p_{\boldsymbol{\varphi}_{N_\infty}}.$$

But Equation (5.9) implies directly that $\boldsymbol{v}_{\boldsymbol{\psi}_{D_\infty}} = \boldsymbol{v}_{\boldsymbol{\varphi}_{N_\infty}} + \boldsymbol{c}$, with $\boldsymbol{c} \in \mathbb{R}^N$, and passing to the limit in the first equations in each Stokes problem of (5.8), we get $p_{\psi_{D_\infty}} = p_{\varphi_{N_\infty}} + c$, with $c \in \mathbb{R}$. In particular, passing to the limit gives $\boldsymbol{v}_{\boldsymbol{\varphi}_{N_\infty}} = \boldsymbol{g}_D$ and $\sigma(\boldsymbol{v}_{\boldsymbol{\varphi}_{N_\infty}}, p_{\boldsymbol{\psi}_{D_\infty}})\mathbf{n} = \sigma(\boldsymbol{v}_{\boldsymbol{\psi}_{D_\infty}}, p_{\boldsymbol{\psi}_{D_\infty}})\mathbf{n} = \boldsymbol{g}_N$ on Γ_{obs} by weak continuity of the trace and normal derivative on Γ_{obs}. Therefore $(\boldsymbol{v}_{\boldsymbol{\varphi}_{N_\infty}}, p_{\boldsymbol{\psi}_{D_\infty}}) = (\boldsymbol{v}, p)$. Hence, we have the following weak convergences in $\mathbf{H}^2(\Omega)$

$$\boldsymbol{v}_{\boldsymbol{\psi}_D^m} \rightharpoonup \boldsymbol{v} + \boldsymbol{c}, \quad \boldsymbol{v}_{\boldsymbol{\varphi}_N^m} \rightharpoonup \boldsymbol{v}$$

and the following weak convergences in $\mathrm{H}^1(\Omega)$

$$p_{\boldsymbol{\psi}_D^m} \rightharpoonup p, \quad p_{\boldsymbol{\varphi}_N^m} \rightharpoonup p + c.$$

Now, we see that $F(\boldsymbol{\varphi}_N^{\rm ex}, \boldsymbol{\psi}_D^{\rm ex}) = 0 = F(\boldsymbol{\varphi}_N^{\rm ex} + c\,\mathbf{n}, \boldsymbol{\psi}_D^{\rm ex} + \boldsymbol{c})$ for any $c \in \mathbb{R}$ and $\boldsymbol{c} \in \mathbb{R}^N$. Therefore, similarly has previously, we have $F_\varepsilon(\boldsymbol{\varphi}_N^m, \boldsymbol{\psi}_D^m) \le F_\varepsilon(\boldsymbol{\varphi}_N^{\rm ex} + c\,\mathbf{n}, \boldsymbol{\psi}_D^{\rm ex} + \boldsymbol{c})$ which implies

$$\begin{aligned}\|(\boldsymbol{v}_{\boldsymbol{\varphi}_N^\varepsilon}, p_{\boldsymbol{\varphi}_N^\varepsilon})\|^2_{\mathbf{H}^2(\Omega)\times\mathrm{H}^1(\Omega)} + \|(\boldsymbol{v}_{\boldsymbol{\psi}_D^\varepsilon}, p_{\boldsymbol{\psi}_D^\varepsilon})\|^2_{\mathbf{H}^2(\Omega)\times\mathrm{H}^1(\Omega)} \le \|(\boldsymbol{v}, p + c)\|^2_{\mathbf{H}^2(\Omega)\times\mathrm{H}^1(\Omega)} \\ + \|(\boldsymbol{v} + \boldsymbol{c}, p)\|^2_{\mathbf{H}^2(\Omega)\times\mathrm{H}^1(\Omega)}\end{aligned}$$

directly implying that the weak convergences are actually strong convergences.

Finally, a standard argument *ad absurdum* ends the proof as in the end of the proof of Theorem 5.2. ☐

We now prove Theorem 1.6 for this penalized Kohn-Vogelius method, recalling that $(\boldsymbol{v}_\varepsilon, p_\varepsilon) \overset{\rm def}{=} (\boldsymbol{v}_{\boldsymbol{\varphi}_N^\varepsilon}, p_{\boldsymbol{\psi}_D^\varepsilon})$.

Proof of Theorem 1.6 for the penalized Kohn-Vogelius method. It is not difficult to verify that we have the *a priori* bounds (see (5.10))

$$\|\boldsymbol{v}_\varepsilon - \boldsymbol{v}\|_{\mathbf{H}^2(\Omega)} \le C(\boldsymbol{v}, p), \quad \|p_\varepsilon - p\|_{\mathrm{H}^1(\Omega)} \le \widetilde{C}(\boldsymbol{v}, p)$$

where $C(\boldsymbol{v}, p)$ and $\widetilde{C}(\boldsymbol{v}, p)$ are constants depending only on the $\mathbf{H}^2(\Omega) \times \mathrm{H}^1(\Omega)$ norm of $(\boldsymbol{v}, p)$. Furthermore, thanks to (5.9), we see that

$$\begin{aligned}\|\sigma(\boldsymbol{v}_\varepsilon, p_\varepsilon)\mathbf{n} - \boldsymbol{g}_N\|_{\mathbf{H}^{1/2}(\Gamma_{\rm obs})} &= \|\sigma(\boldsymbol{v}_{\boldsymbol{\varphi}_N^\varepsilon}, p_{\boldsymbol{\psi}_D^\varepsilon})\mathbf{n} - \sigma(\boldsymbol{v}_{\boldsymbol{\psi}_D^\varepsilon}, p_{\boldsymbol{\psi}_D^\varepsilon})\mathbf{n}\|_{\mathbf{H}^{1/2}(\Gamma_{\rm obs})} \\ &\le |\boldsymbol{v}_\varepsilon - \boldsymbol{v}_{\boldsymbol{\psi}_D^\varepsilon}|_{\mathbf{H}^2(\Omega)} + |\boldsymbol{v}_\varepsilon - \boldsymbol{v}_{\boldsymbol{\psi}_D^\varepsilon}|_{\mathbf{H}^1(\Omega)} \qquad (5.11) \\ &\le \sqrt{\varepsilon}\, C(\boldsymbol{v}, p)\end{aligned}$$

where $C(\boldsymbol{v}, p)$ is another constant depending only on $\mathbf{H}^2(\Omega) \times \mathrm{H}^1(\Omega)$ norm of $(\boldsymbol{v}, p)$.

Hence, applying again estimates (1.5) and (1.6), we directly obtain the announced result. ☐

Remark 5.7. *The Kohn-Vogelius functional is classically defined by* $\mathcal{F}(\boldsymbol{\varphi}_N, \boldsymbol{\psi}_D) = |\boldsymbol{v}_{\boldsymbol{\varphi}_N} - \boldsymbol{v}_{\boldsymbol{\psi}_D}|^2_{\mathbf{H}^1(\Omega)}$ *instead of* $F(\boldsymbol{\varphi}_N, \boldsymbol{\psi}_D) = |\boldsymbol{v}_{\boldsymbol{\varphi}_N} - \boldsymbol{v}_{\boldsymbol{\psi}_D}|^2_{\mathbf{H}^2(\Omega)} + |\boldsymbol{v}_{\boldsymbol{\varphi}_N} - \boldsymbol{v}_{\boldsymbol{\psi}_D}|^2_{\mathbf{H}^1(\Omega)}$. *Notice that Proposition 5.5 is also valid for the associated functional* $\mathcal{F}_\varepsilon$. *The only point where the* $\mathbf{H}^2$*-seminorm is needed is Inequality* (5.11).

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