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THE “EXTERIOR APPROACH” TO SOLVE THE INVERSE OBSTACLE PROBLEM FOR THE STOKES SYSTEM

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Abstract. We apply an “exterior approach” based on the coupling of a method of quasi-reversibility and of a level set method in order to recover a fixed obstacle immersed in a Stokes flow from boundary measurements. Concerning the method of quasi-reversibility, two new mixed formulations are introduced in order to solve the ill-posed Cauchy problems for the Stokes system by using some classical conforming finite elements. We provide some proofs for the convergence of the quasi-reversibility methods on the one hand and of the level set method on the other hand. Some numerical experiments in $2D$ show the efficiency of the two mixed formulations and of the exterior approach based on one of them.

1. Introduction

In this paper we address the inverse obstacle problem for the Stokes system. Such problem consists in finding a fixed obstacle immersed in a fluid from boundary measurements. Here we only consider the stationary case, which has been treated by several authors in the past. The fundamental result that makes such problem solvable is unique continuation for the Stokes system. Such result is a straightforward consequence of unique continuation for the Laplace equation. Note however that if the model involves a potential, the unique continuation property is much more difficult to obtain, in particular when such potential is not regular, as can be seen in [1]. The stability issue for the ill-posed Cauchy problem is studied in [2], where in particular a three-ball inequality is obtained, and in [3], where a stability estimate is obtained in two dimensions by using a Carleman estimate borrowed from [4] and by following the method of [5]. Concerning the regularization of the ill-posed Cauchy problem, a method based on the minimization of the Kohn-Vogelius functional is applied in [6]. From the unique continuation result, it is easy to prove the uniqueness property for the inverse obstacle problem [7], which paves the way to study some effective identification methods. Several ones were applied in the literature, for example optimization methods based on a parametrization of the obstacle [8], optimization methods based on shape derivative [9, 10], and topological gradient methods [11, 12].

In the following paper we propose to apply the “exterior approach” which is introduced in [13, 14] for the Laplace equation in order to solve the inverse obstacle problem for the Stokes system. The obstacle is characterized by a homogeneous

Dirichlet boundary condition. The exterior approach consists in defining a decreasing sequence of domains that converge to the obstacle in the sense of Hausdorff distance. More precisely, such iterative approach is based on a combination of a quasi-reversibility method to update the solution of the ill-posed Cauchy problem outside the obstacle obtained at previous iteration and of a level set method to update the obstacle with the help of the solution obtained at previous iteration. In some sense, the exterior approach consists, the inverse obstacle problem being ill-posed (because of the nature of data) and non-linear (because it amounts to a free boundary problem), in treating separately the illposedness and the nonlinearity. Concerning the quasi-reversibility method, in order to use standard finite element methods, we introduce two different mixed formulations to regularize the ill-posed Stokes system. They can be used in another context as the inverse obstacle problem, and since they have different advantages depending on the situation, we describe both of them. Concerning the level set method, we use the method based on a simple Poisson problem already introduced in [13] for the Laplace equation. Here we simply adapt its justification to the framework of the Stokes system. Before introducing the different sections of the paper, we should insist on the fact that in what follows the obstacle is fixed. The more complicated case of a moving obstacle has recently been addressed in [15, 16, 17].

Our paper is organized as follows. In section 2 we introduce the inverse obstacle problem, and recall in particular the uniqueness result. Two different formulations of the quasi-reversibility method are then exposed and justified in section 3. In section 4 we present and justify our level set method, which is then included in the global algorithm of the exterior approach to solve the inverse obstacle problem. Section 5 introduces the finite element discretizations that are associated with each mixed formulation of quasi-reversibility. Lastly some numerical results are shown in section 6 : firstly the two methods of quasi-reversibility are compared on a simple Cauchy problem, secondly one of them is chosen and some examples of reconstruction with the help of our exterior approach are presented.

2. STATEMENT OF THE INVERSE PROBLEM

The inverse obstacle problem for the Stokes system is defined as follows. Let be an open, bounded and connected domain of $\mathbb{R}^d$, $d \geq 2$, with Lipschitz boundary. Let $O \Subset$ be another open domain with a Lipschitz boundary, referred to as the obstacle, and such that $\Omega := \quad \setminus \overline{O}$ is connected. Note that O is not necessarily connected. Let Γ be an open and nonempty subset of ∂ . Given a pair of data $(\boldsymbol{g}_0, \boldsymbol{g}_1) \in (H^{1/2}(\Gamma))^d \times (H^{-1/2}(\Gamma))^d$ with $\boldsymbol{g}_0 \neq 0$, the inverse obstacle problem consists in finding a domain O and some functions $\boldsymbol{u} \in (H^1(\Omega))^d$ and $p \in \ ^2(\Omega)$ which satisfy

$$(1) \qquad \begin{cases} -\nu\Delta \boldsymbol{u} + \nabla p & = \boldsymbol{0} & \text{in } \Omega \\ \operatorname{div} \boldsymbol{u} & = 0 & \text{in } \Omega \\ \boldsymbol{u} & = \boldsymbol{g}_0 & \text{on } \Gamma \\ 2\nu e(\boldsymbol{u}) \cdot \boldsymbol{n} - p\boldsymbol{n} & = \boldsymbol{g}_1 & \text{on } \Gamma \\ \boldsymbol{u} & = \boldsymbol{0} & \text{on } \partial O, \end{cases}$$

where $\boldsymbol{n}$ is the outward unit normal, the function $\boldsymbol{u}$ is the velocity field, the function p is the pressure field, the strain field $e(\boldsymbol{u})$ is defined by

$$e(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla^t \boldsymbol{u}),$$

and $\nu > 0$ is the kinematic viscosity of the fluid.

The second boundary condition on Γ means that the normal stress $\sigma \cdot \boldsymbol{n}$ is given by $\boldsymbol{g_1}$ on Γ, since the stress field is given by

$$\sigma = 2\nu e(\boldsymbol{u}) - p I,$$

where I is the identity matrix of size d. We first recall the following uniqueness result concerning our inverse obstacle problem, which is proved in [7].

Proposition 1. *Let two domains O^1, O^2 and corresponding pairs of functions $(\boldsymbol{u}^1, p^1)$, $(\boldsymbol{u}^2, p^2)$ satisfy problem (1) with data $(\boldsymbol{g}_0, \boldsymbol{g}_1)$, $\boldsymbol{g}_0 \neq 0$. Assume in addition that $\boldsymbol{u}^1$ and $\boldsymbol{u}^2$ are continuous fields in Ω^1 and Ω^2, respectively, up to the boundary. Then we have $O^1 = O^2$ and $(\boldsymbol{u}^1, p^1) = (\boldsymbol{u}^2, p^2)$.*

For the readers convenience we recall the proof of such result, which is based on the following straightforward consequence of unique continuation.

Lemma 2.1. *Assume $(\boldsymbol{u}, p) \in (H^1(\Omega))^d \times L^2(\Omega)$ satisfies*

$$\begin{cases} -\nu\Delta\boldsymbol{u} + \nabla p &= \boldsymbol{0} \quad \text{in } \Omega \\ \operatorname{div}\boldsymbol{u} &= 0 \quad \text{in } \Omega \\ \boldsymbol{u} &= \boldsymbol{0} \quad \text{on } \Gamma \\ 2\nu e(\boldsymbol{u}) \cdot \boldsymbol{n} - p\boldsymbol{n} &= \boldsymbol{0} \quad \text{on } \Gamma, \end{cases}$$

then $(\boldsymbol{u}, p) = (\boldsymbol{0}, 0)$.

Proof. Let us consider $\tilde{\Omega}$ the connected component of $\mathbb{R}^d \setminus \overline{O^1 \cup O^2}$ which is in contact with Γ, and $\tilde{O} := \mathbb{R}^d \setminus \overline{\tilde{\Omega}}$. Consider $\boldsymbol{u} := \boldsymbol{u}^1 - \boldsymbol{u}^2$ and $p := p^1 - p^2$. The pair $(\boldsymbol{u}, p)$ satisfies in $\tilde{\Omega}$ the system

$$\begin{cases} -\nu\Delta\boldsymbol{u} + \nabla p &= \boldsymbol{0} \quad \text{in } \tilde{\Omega} \\ \operatorname{div}\boldsymbol{u} &= 0 \quad \text{in } \tilde{\Omega} \\ \boldsymbol{u} &= \boldsymbol{0} \quad \text{on } \Gamma \\ 2\nu e(\boldsymbol{u}) \cdot \boldsymbol{n} - p\boldsymbol{n} &= \boldsymbol{0} \quad \text{on } \Gamma. \end{cases}$$

By using the lemma 2.1, we obtain that $(\boldsymbol{u}, p) = (\boldsymbol{0}, 0)$ in $\tilde{\Omega}$, and in particular $\boldsymbol{u}^1 = \boldsymbol{u}^2$ on $\partial\tilde{O}$. Let us consider now the open domain $\tilde{O} \setminus \overline{O^2}$. We have $\boldsymbol{u}^2 = \boldsymbol{0}$ on $\partial(\tilde{O} \setminus \overline{O^2})$ and since $\boldsymbol{u}^2$ is continuous in the open domain $\tilde{O} \setminus \overline{O^2}$ up to the boundary, from [18] (see theorem IX.17 and remark 20) we obtain that $\boldsymbol{u}^2 \in (H_0^1(\tilde{O} \setminus \overline{O^2}))^d$. If we consider a function $\boldsymbol{\phi}^2 \in (C_0^\infty(\tilde{O} \setminus \overline{O^2}))^d$, since $-\nu\Delta\boldsymbol{u}^2 + \nabla p^2 = \boldsymbol{0}$ in $\tilde{O} \setminus \overline{O^2}$ in the distributional sense, we have

$$\sum_{i,j=1}^{d} \left\langle -\nu \frac{\partial^2 \boldsymbol{u}_j^2}{\partial x_i^2} + \frac{\partial p^2}{\partial x_j}, \boldsymbol{\phi}_j^2 \right\rangle = 0 = \int_{\tilde{O}\setminus\overline{O^2}} \left(\nu \sum_{i,j=1}^{d} \frac{\partial \boldsymbol{u}_j^2}{\partial x_i} \frac{\partial \boldsymbol{\phi}_j^2}{\partial x_i} - p^2 \frac{\partial \boldsymbol{\phi}_j^2}{\partial x_j} \right) dx$$

and by density of $(C_0^\infty(\tilde{O} \setminus \overline{O^2}))^d$ in $(H_0^1(\tilde{O} \setminus \overline{O^2}))^d$,

$$\int_{\tilde{O}\setminus\overline{O^2}} (\nu|\nabla\boldsymbol{u}^2|^2 - p^2 \operatorname{div}\boldsymbol{u}^2)\, dx = 0,$$

where we have denoted

$$|\nabla\boldsymbol{u}|^2 = \sum_{i,j=1}^{d} \left(\frac{\partial \boldsymbol{u}_j}{\partial x_i}\right)^2.$$

We conclude that $\nabla \boldsymbol{u}$

The spaces V_0 and $\tilde{V}_0$ are endowed with the scalar product

$$(\boldsymbol{u},\boldsymbol{v})_V = \int_\Omega e(\boldsymbol{u}) : e(\boldsymbol{v})\,dx = \sum_{i,j=1}^{d} \int_\Omega e_{ij}(\boldsymbol{u}) e_{ij}(\boldsymbol{v})\,dx,$$

the associated norm $||\cdot||_V$ being equivalent to the standard norm of V with the help of Korn's inequality [23]. We consider the following weak formulation $(P_{\varepsilon,\gamma})$ for $\varepsilon, \gamma > 0$: find $(\boldsymbol{u}_{\varepsilon,\gamma}, \boldsymbol{\lambda}_{\varepsilon,\gamma}) \in V_g \times \tilde{V}_0$ such that

$$(3) \qquad \left\{ \begin{array}{l} 2\eta\varepsilon \displaystyle\int_\Omega e(\boldsymbol{u}_{\varepsilon,\gamma}) : e(\boldsymbol{v})\,dx + \int_\Omega \mathrm{div}\boldsymbol{u}_{\varepsilon,\gamma}\,\mathrm{div}\boldsymbol{v}\,dx \\ \qquad + 2\eta \displaystyle\int_\Omega e(\boldsymbol{v}) : e(\boldsymbol{\lambda}_{\varepsilon,\gamma})\,dx = 0, \quad \forall \boldsymbol{v} \in V_0 \\ 2\eta \displaystyle\int_\Omega e(\boldsymbol{u}_{\varepsilon,\gamma}) : e(\boldsymbol{\mu})\,dx - \frac{1}{\varepsilon}\int_\Omega \mathrm{div}\boldsymbol{\lambda}_{\varepsilon,\gamma}\,\mathrm{div}\boldsymbol{\mu}\,dx \\ -\gamma \displaystyle\int_\Omega e(\boldsymbol{\lambda}_{\varepsilon,\gamma}) : e(\boldsymbol{\mu})\,dx = \int_{\Gamma_1} \boldsymbol{g_1}\cdot\boldsymbol{\mu}\,ds, \quad \forall \boldsymbol{\mu} \in \tilde{V}_0\,. \end{array} \right.$$

Remark 1. We remark that problem $(P_{\varepsilon,\gamma})$ is equivalent to : find $(\boldsymbol{u}_{\varepsilon,\gamma}, \boldsymbol{\lambda}_{\varepsilon,\gamma}, p_{\varepsilon,\gamma}) \in V_g \times \tilde{V}_0 \times L^2(\Omega)$ such that

$$(4) \qquad \left\{ \begin{array}{l} 2\varepsilon\eta \displaystyle\int_\Omega e(\boldsymbol{u}_{\varepsilon,\gamma}) : e(\boldsymbol{v})\,dx + \int_\Omega \mathrm{div}\boldsymbol{u}_{\varepsilon,\gamma}\,\mathrm{div}\boldsymbol{v}\,dx \\ \qquad + 2\eta \displaystyle\int_\Omega e(\boldsymbol{v}) : e(\boldsymbol{\lambda}_{\varepsilon,\gamma})\,dx = 0, \quad \forall \boldsymbol{v} \in V_0\,, \\ \qquad 2\eta \displaystyle\int_\Omega e(\boldsymbol{u}_{\varepsilon,\gamma}) : e(\boldsymbol{\mu})\,dx - \int_\Omega p_{\varepsilon,\gamma}\,\mathrm{div}\boldsymbol{\mu}\,dx \\ -\gamma \displaystyle\int_\Omega e(\boldsymbol{\lambda}_{\varepsilon,\gamma}) : e(\boldsymbol{\mu})\,dx = \int_{\Gamma_1} \boldsymbol{g_1}\cdot\boldsymbol{\mu}\,ds, \quad \forall \boldsymbol{\mu} \in \tilde{V}_0\,, \\ \varepsilon \displaystyle\int_\Omega p_{\varepsilon,\gamma}\,q\,dx - \int_\Omega q\,\mathrm{div}\boldsymbol{\lambda}_{\varepsilon,\gamma}\,dx = 0, \quad \forall q \in L^2(\Omega). \end{array} \right.$$

Such modified formulation involves the approximate pressure field $p_{\varepsilon,\gamma}$, contrary to formulation (3).

We have the following theorem.

Theorem 3.1. *The problem $(P_{\varepsilon,\gamma})$ given by (3) has a unique solution $(\boldsymbol{u}_{\varepsilon,\gamma}, \boldsymbol{\lambda}_{\varepsilon,\gamma}) \in V_g \times \tilde{V}_0$. If the data $(\boldsymbol{g_0}, \boldsymbol{g_1})$ is such that the Cauchy problem (2) has a (unique) solution $(\boldsymbol{u}, p)$ in $V \times L^2(\Omega)$ and if γ is a bounded function of ε such that*

$$\lim_{\varepsilon\to 0} \varepsilon\,\gamma(\varepsilon) = 0,$$

then

$$\lim_{\varepsilon\to 0} (\boldsymbol{u}_{\varepsilon,\gamma(\varepsilon)}, \boldsymbol{\lambda}_{\varepsilon,\gamma(\varepsilon)}, p_{\varepsilon,\gamma(\varepsilon)}) = (\boldsymbol{u}, \boldsymbol{0}, p) \quad \text{in} \quad V \times V \times L^2(\Omega),$$

where

$$p_{\varepsilon,\gamma} = \mathrm{div}\boldsymbol{\lambda}_{\varepsilon,\gamma}/\varepsilon,$$

and we have the estimates

$$||\mathrm{div}\boldsymbol{u}_{\varepsilon,\gamma}||_{L^2(\Omega)} \le \sqrt{\varepsilon}\,\delta \quad , \quad ||\boldsymbol{\lambda}_{\varepsilon,\gamma}||_V \le \sqrt{\varepsilon\,\gamma(\varepsilon)}\,\delta$$

with

$$\delta := \sqrt{2\eta||\boldsymbol{u}||_V^2 + ||p||^2_{L^2(\Omega)}}.$$

Proof. Considering some element $\boldsymbol{U} \in V_g$ and defining the new variable $\hat{\boldsymbol{u}}_{\varepsilon,\gamma} := \boldsymbol{u}_{\varepsilon,\gamma} - \boldsymbol{U}$, the problem $(\mathcal{P}_{\varepsilon,\gamma})$ is equivalent to find $(\hat{\boldsymbol{u}}_{\varepsilon,\gamma}, \boldsymbol{\lambda}_{\varepsilon,\gamma}) \in V_0 \times \tilde{V}_0$ such that for all $(\boldsymbol{v}, \boldsymbol{\mu}) \in V_0 \times \tilde{V}_0$,

$$A_{\varepsilon,\gamma}\left((\hat{\boldsymbol{u}}_{\varepsilon,\gamma}, \boldsymbol{\lambda}_{\varepsilon,\gamma}); (\boldsymbol{v}, \boldsymbol{\mu})\right) = -A_{\varepsilon,\gamma}\left((\boldsymbol{U}, \boldsymbol{0}); (\boldsymbol{v}, \boldsymbol{\mu})\right) - \int \boldsymbol{g_1} \cdot \boldsymbol{\mu}\, ds,$$

with

$$\begin{aligned} &A_{\varepsilon,\gamma}\left((\boldsymbol{u}, \boldsymbol{\lambda}); (\boldsymbol{v}, \boldsymbol{\mu})\right) = \\ &\int_\Omega (\mathrm{div}\boldsymbol{u})(\mathrm{div}\boldsymbol{v})\, dx + \frac{1}{\varepsilon}\int_\Omega (\mathrm{div}\boldsymbol{\lambda})(\mathrm{div}\boldsymbol{\mu})\, dx \\ &+2\nu \int_\Omega e(\boldsymbol{v}) : e(\boldsymbol{\lambda})\, dx - 2\nu \int_\Omega e(\boldsymbol{u}) : e(\boldsymbol{\mu})\, dx \\ &+2\varepsilon\nu \int_\Omega e(\boldsymbol{u}) : e(\boldsymbol{v})\, dx + \gamma \int_\Omega e(\boldsymbol{\lambda}) : e(\boldsymbol{\mu})\, dx. \end{aligned}$$

Well-posedness of problem $(\mathcal{P}_{\varepsilon,\gamma})$ is then an obvious consequence of Lax-Milgram lemma since for all $(\boldsymbol{v}, \boldsymbol{\mu}) \in V_0 \times \tilde{V}_0$,

$$A_{\varepsilon,\gamma}\left((\boldsymbol{v}, \boldsymbol{\mu}); (\boldsymbol{v}, \boldsymbol{\mu})\right) \geq 2\varepsilon\nu ||\boldsymbol{v}||_V^2 + \gamma ||\boldsymbol{\mu}||_V^2. \tag{5}$$

We now consider the second part of the proposition. For sake of simplicity the solution $\boldsymbol{u}_{\varepsilon,\gamma(\varepsilon)}$ is denoted $\boldsymbol{u}_\varepsilon$, and we do the same for $\boldsymbol{\lambda}_{\varepsilon,\gamma(\varepsilon)}$ and $p_{\varepsilon,\gamma(\varepsilon)}$. From remark 1 we have

$$\left\{ \begin{aligned} &2\varepsilon\nu \int_\Omega e(\boldsymbol{u}_\varepsilon) : e(\boldsymbol{v})\, dx + \int_\Omega \mathrm{div}\boldsymbol{u}_\varepsilon\, \mathrm{div}\boldsymbol{v}\, dx \\ &\quad +2\nu \int_\Omega e(\boldsymbol{v}) : e(\boldsymbol{\lambda}_\varepsilon)\, dx = 0, \quad \forall \boldsymbol{v} \in V_0, \\ &2\nu \int_\Omega e(\boldsymbol{u}_\varepsilon) : e(\boldsymbol{\mu})\, dx - \int_\Omega p_\varepsilon\, \mathrm{div}\boldsymbol{\mu}\, dx \\ &\quad -\gamma(\varepsilon) \int_\Omega e(\boldsymbol{\lambda}_\varepsilon) : e(\boldsymbol{\mu})\, dx = \int \boldsymbol{g_1} \cdot \boldsymbol{\mu}\, ds, \quad \forall \boldsymbol{\mu} \in \tilde{V}_0, \\ &\varepsilon \int_\Omega p_\varepsilon q\, dx - \int_\Omega q\, \mathrm{div}\boldsymbol{\lambda}_\varepsilon\, dx = 0, \quad \forall q \in L^2(\Omega). \end{aligned} \right. \tag{6}$$

Secondly we remark that the equations $-\nu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{0}$ in Ω, $\mathrm{div}\boldsymbol{u} = 0$ in Ω and $2\nu e(\boldsymbol{u}) \cdot \boldsymbol{n} - p\boldsymbol{n} = \boldsymbol{g_1}$ on Γ satisfied by the exact solution $(\boldsymbol{u}, p)$ imply that

$$2\nu \int_\Omega e(\boldsymbol{u}) : e(\boldsymbol{\mu})\, dx - \int_\Omega p\, \mathrm{div}\boldsymbol{\mu}\, dx = \int \boldsymbol{g_1} \cdot \boldsymbol{\mu}\, ds, \quad \forall \boldsymbol{\mu} \in \tilde{V}_0.$$

Here we have used the fact that since $\mathrm{div}\boldsymbol{u} = 0$, $\Delta\boldsymbol{u} = 2\,\mathrm{div}e(\boldsymbol{u})$. By subtracting such equation to the second equation of (6) we obtain

$$2\nu \int_\Omega e(\boldsymbol{u}_\varepsilon - \boldsymbol{u}) : e(\boldsymbol{\mu})\, dx - \int_\Omega (p_\varepsilon - p)\, \mathrm{div}\boldsymbol{\mu}\, dx - \gamma(\varepsilon) \int_\Omega e(\boldsymbol{\lambda}_\varepsilon) : e(\boldsymbol{\mu})\, dx = 0, \quad \forall \boldsymbol{\mu} \in \tilde{V}_0.$$

Choosing now $\boldsymbol{v} = \boldsymbol{u}_\varepsilon - \boldsymbol{u}$ in the first equation of (6), $q = p_\varepsilon - p$ in the third equation of (6) and $\boldsymbol{\mu} = \boldsymbol{\lambda}_\varepsilon$ in the above equation, we obtain

$$\left\{ \begin{aligned} 2\varepsilon\nu \int_\Omega e(\boldsymbol{u}_\varepsilon) : e(\boldsymbol{u}_\varepsilon - \boldsymbol{u})\, dx + \int_\Omega (\mathrm{div}\boldsymbol{u}_\varepsilon)^2\, dx + 2\nu \int_\Omega e(\boldsymbol{u}_\varepsilon - \boldsymbol{u}) : e(\boldsymbol{\lambda}_\varepsilon)\, dx &= 0 \\ -2\nu \int_\Omega e(\boldsymbol{u}_\varepsilon - \boldsymbol{u}) : e(\boldsymbol{\lambda}_\varepsilon)\, dx + \int_\Omega (p_\varepsilon - p)\, \mathrm{div}\boldsymbol{\lambda}_\varepsilon\, dx + \gamma(\varepsilon) \int_\Omega e(\boldsymbol{\lambda}_\varepsilon) : e(\boldsymbol{\lambda}_\varepsilon)\, dx &= 0 \\ \varepsilon \int_\Omega p_\varepsilon (p_\varepsilon - p)\, dx - \int_\Omega (p_\varepsilon - p)\, \mathrm{div}\boldsymbol{\lambda}_\varepsilon\, dx &= 0. \end{aligned} \right.$$

Adding the three above equations, we obtain

$$2\varepsilon\mu(\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon - \boldsymbol{u})_V + \varepsilon(p_\varepsilon, p_\varepsilon - p)_{L^2(\Omega)} + ||\mathrm{div}\boldsymbol{u}_\varepsilon||^2_{L^2(\Omega)} + \gamma(\varepsilon)||\boldsymbol{\lambda}_\varepsilon||^2_V = 0. \tag{7}$$

From equation (7) we first obtain

$$2\mu||\boldsymbol{u}_\varepsilon||^2_V + ||p_\varepsilon||^2_{L^2(\Omega)} \leq 2\mu||\boldsymbol{u}||^2_V + ||p||^2_{L^2(\Omega)},$$

that is $\boldsymbol{u}_\varepsilon$ and p_ε are bounded. Hence we can extract subsequences $\boldsymbol{u}_\varepsilon$ and p_ε with $\boldsymbol{u}_\varepsilon \rightharpoonup \boldsymbol{w}$ in V_g and $p_\varepsilon \rightharpoonup \pi$ in $L^2(\Omega)$. It is clear that $\mathrm{div}\boldsymbol{u}_\varepsilon \rightharpoonup \mathrm{div}\boldsymbol{w}$ in $L^2(\Omega)$. By subtracting $2\mu\varepsilon(\boldsymbol{u}, \boldsymbol{u}_\varepsilon - \boldsymbol{u})_V + \varepsilon(p, p_\varepsilon - p)_{L^2(\Omega)}$ to equation (7), we also obtain

$$2\mu||\boldsymbol{u}_\varepsilon - \boldsymbol{u}||^2_V + ||p_\varepsilon - p||^2_{L^2(\Omega)} \leq -2\mu(\boldsymbol{u}, \boldsymbol{u}_\varepsilon - \boldsymbol{u})_V - (p, p_\varepsilon - p)_{L^2(\Omega)}, \tag{8}$$

so that

$$2\mu||\boldsymbol{u}_\varepsilon - \boldsymbol{u}||^2_V + ||p_\varepsilon - p||^2_{L^2(\Omega)} \leq 2\mu||\boldsymbol{u}||^2_V + ||p||^2_{L^2(\Omega)},$$

and then

$$\gamma(\varepsilon)||\boldsymbol{\lambda}_\varepsilon||^2_V \leq \varepsilon(2\mu||\boldsymbol{u}||^2_V + ||p||^2_{L^2(\Omega)}).$$

The fact that $\varepsilon/\gamma(\varepsilon) \to 0$ when $\varepsilon \to 0$ implies that $\boldsymbol{\lambda}_\varepsilon \to \boldsymbol{0}$ in V. Passing to the limit in the first equation of (6), we obtain

$$\int_\Omega \mathrm{div}\boldsymbol{w}\,\mathrm{div}\boldsymbol{v}\,dx = 0, \quad \forall \boldsymbol{v} \in V_0.$$

It is straightforward to prove that the operator $\mathrm{div} : V_0 \to L^2(\Omega)$ has dense range, we then conclude that $\mathrm{div}\boldsymbol{w} = 0$ in Ω. Passing to the limit in the second equation of (6) and using the fact that $\gamma(\varepsilon)$ is bounded, we obtain

$$2\mu\int_\Omega e(\boldsymbol{w}) : e(\boldsymbol{\mu}) - \int_\Omega \pi\,\mathrm{div}\boldsymbol{\mu} = \int \boldsymbol{g_1}\cdot\boldsymbol{\mu}\,ds, \quad \forall \boldsymbol{\mu} \in \tilde{V}_0,$$

that is $-\mu\Delta\boldsymbol{w} + \nabla\pi = \boldsymbol{0}$ in Ω and $2\mu e(\boldsymbol{w})\cdot\boldsymbol{n} - \pi\boldsymbol{n} = \boldsymbol{g_1}$ on Γ. As a conclusion, $(\boldsymbol{w}, \pi) \in V \times L^2(\Omega)$ solves the Cauchy problem (2) and from lemma 2.1 we obtain $(\boldsymbol{w}, \pi) = (\boldsymbol{u}, p)$. From equation (8), clearly weak convergence implies strong convergence. A classical contradiction argument enables us to conclude that the whole sequences $\boldsymbol{u}_\varepsilon$ and p_ε converge to $\boldsymbol{u}$ and p in V and $L^2(\Omega)$, respectively.
Lastly, the estimates of $\mathrm{div}\boldsymbol{u}_\varepsilon$ and $\boldsymbol{\lambda}_\varepsilon$ follow from (7). □

We now look at the case when the data $(\boldsymbol{g_0}, \boldsymbol{g_1})$ are incompatible, that is the Cauchy problem (2) has no solution.

Theorem 3.2. *If the data $(\boldsymbol{g_0}, \boldsymbol{g_1})$ is such that the Cauchy problem (2) has no solution, then if the function $\gamma(\varepsilon)$ satisfies the same assumptions as in theorem 3.1, we have*

$$\lim_{\varepsilon\to 0}\left\{||\boldsymbol{u}_{\varepsilon,\gamma(\varepsilon)}||_V + ||\boldsymbol{\lambda}_{\varepsilon,\gamma(\varepsilon)}||_V + ||(\mathrm{div}\boldsymbol{\lambda}_{\varepsilon,\gamma(\varepsilon)})/\sqrt{\varepsilon}||_{L^2(\Omega)}\right\} = +\infty. \tag{9}$$

Proof. We reuse the short notation $(\boldsymbol{u}_{\varepsilon,\gamma(\varepsilon)}, \boldsymbol{\lambda}_{\varepsilon,\gamma(\varepsilon)}) = (\boldsymbol{u}_\varepsilon, \boldsymbol{\lambda}_\varepsilon)$. By contradiction, if (9) is not true, it means that we can extract from $(\boldsymbol{u}_\varepsilon, \boldsymbol{\lambda}_\varepsilon) \in V_g \times \tilde{V}_0$ a bounded subsequence, still denoted $(\boldsymbol{u}_\varepsilon, \boldsymbol{\lambda}_\varepsilon)$, such that $\mathrm{div}\boldsymbol{\lambda}_\varepsilon/\sqrt{\varepsilon}$ is bounded in $L^2(\Omega)$. From that sequence we extract again a subsequence such that $\boldsymbol{u}_\varepsilon \rightharpoonup \boldsymbol{w}$ in V_g, $\boldsymbol{\lambda}_\varepsilon \rightharpoonup \boldsymbol{l}$ in $\tilde{V}_0$ and $\mathrm{div}\boldsymbol{\lambda}_\varepsilon/\sqrt{\varepsilon} \rightharpoonup \pi$ in $L^2(\Omega)$. This implies that $\mathrm{div}\boldsymbol{u}_\varepsilon \rightharpoonup -\rho := \mathrm{div w}$ and $\mathrm{div}\boldsymbol{\lambda}_\varepsilon \rightharpoonup \mathrm{div}\boldsymbol{l}$ in $L^2(\Omega)$. Since $\mathrm{div}\boldsymbol{\lambda}_\varepsilon \to 0$ in $L^2(\Omega)$, we have in particular $\mathrm{div}\boldsymbol{l} = 0$. By passing to the limit in the first equation of (3), we obtain that

$$2\mu\int_\Omega e(\boldsymbol{v}) : e(\boldsymbol{l})\,dx - \int_\Omega \rho\,\mathrm{div}\boldsymbol{v}\,dx = 0, \quad \forall \boldsymbol{v} \in V_0,$$

which implies (see the proof of theorem 3.1) that $(\boldsymbol{l}, \rho) \in V \times {}^{2}(\Omega)$ solves the Cauchy problem for the Stokes system (2) with vanishing Cauchy data on $\tilde{\Gamma}$ instead of Cauchy data $(\boldsymbol{g}_0, \boldsymbol{g}_1)$ on Γ. From lemma 2.1 we conclude that $(\boldsymbol{l}, \rho) = (\mathbf{0}, 0)$. We now pass to the limit in the second equation of (3) and obtain

$$2\nu \int_\Omega e(\boldsymbol{w}) : e(\boldsymbol{\mu})\, dx - \int_\Omega \pi \operatorname{div} \boldsymbol{\mu}\, dx = \int \boldsymbol{g}_1 \cdot \boldsymbol{\mu}\, ds, \quad \forall \boldsymbol{\mu} \in \tilde{V}_0,$$

which implies that $(\boldsymbol{w}, \pi) \in V \times {}^{2}(\Omega)$ solves the Cauchy problem (2), which is a contradiction. $\ulcorner$

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with

$$\mathbf{M} := \sqrt{||\sigma||^2_\Sigma + ||\boldsymbol{u}||^2_V + ||p||^2_{L^2(\Omega)}}.$$

Proof. Considering some elements $\boldsymbol{\Sigma} \in \Sigma_g$ and $\boldsymbol{U} \in V_g$ and defining the new variables $\hat{\sigma}_\varepsilon := \sigma_\varepsilon - \boldsymbol{\Sigma}$ and $\hat{\boldsymbol{u}}_\varepsilon := \boldsymbol{u}_\varepsilon - \boldsymbol{U}$ the problem (Q_ε) is equivalent to find $(\hat{\sigma}_\varepsilon, \hat{\boldsymbol{u}}_\varepsilon) \in \Sigma_0 \times V_0$ such that for all $(\tau, \boldsymbol{v}) \in \Sigma_0 \times V_0$,

$$B_\varepsilon\left((\hat{\sigma}_\varepsilon, \hat{\boldsymbol{u}}_\varepsilon); (\tau, \boldsymbol{v})\right) = -B_\varepsilon\left((\boldsymbol{\Sigma}, \boldsymbol{U}); (\tau, \boldsymbol{v})\right),$$

with

$$B_\varepsilon\left((\sigma, \boldsymbol{u}); (\tau, \boldsymbol{v})\right) =$$

$$\int_\Omega \left(\mathrm{dev}(\sigma - 2\mu e(\boldsymbol{u})) : \mathrm{dev}(\tau - 2\mu e(\boldsymbol{v})) + \frac{\varepsilon}{d(d+\varepsilon)} \mathrm{tr}(\sigma - 2\mu e(\boldsymbol{u}))\, \mathrm{tr}(\tau - 2\mu e(\boldsymbol{v}))\right) dx$$
$$+ \int_\Omega (\mathrm{div}\sigma) \cdot (\mathrm{div}\tau)\, dx + \int_\Omega (\mathrm{div}\boldsymbol{u})(\mathrm{div}\boldsymbol{v})\, dx$$
$$+ \varepsilon \int_\Omega \left(\sigma : \tau + (\mathrm{div}\sigma) \cdot (\mathrm{div}\tau)\right) dx + \varepsilon \int_\Omega e(\boldsymbol{u}) : e(\boldsymbol{v})\, dx,$$

where for some $d \times d$ matrix s, the deviatoric part of s is defined by $\mathrm{dev}(s) = s - \mathrm{tr}(s) I/d$. This can be seen by using

$$\left(\sigma - 2\mu e(\boldsymbol{u}) - \frac{1}{d+\varepsilon} \mathrm{tr}(\sigma - 2\mu e(\boldsymbol{u})) I\right) : (\tau - 2\mu e(\boldsymbol{v})) =$$

$$\left(\mathrm{dev}(\sigma - 2\mu e(\boldsymbol{u})) + \frac{\varepsilon}{d(d+\varepsilon)} \mathrm{tr}(\sigma - 2\mu e(\boldsymbol{u})) I\right) : \left(\mathrm{dev}(\tau - 2\mu e(\boldsymbol{v})) + \frac{1}{d} \mathrm{tr}(\tau - 2\mu e(\boldsymbol{v})) I\right) =$$

$$\mathrm{dev}(\sigma - 2\mu e(\boldsymbol{u})) : \mathrm{dev}(\tau - 2\mu e(\boldsymbol{v})) + \frac{\varepsilon}{d(d+\varepsilon)} \mathrm{tr}(\sigma - 2\mu e(\boldsymbol{u}))\, \mathrm{tr}(\tau - 2\mu e(\boldsymbol{v})).$$

Well-posedness of problem (Q_ε) is again a consequence of Lax-Milgram lemma since for all $(\tau, \boldsymbol{v}) \in \Sigma_0 \times V_0$,

$$B_\varepsilon\left((\tau, \boldsymbol{v}); (\tau, \boldsymbol{v})\right) \geq \varepsilon ||\tau||^2_\Sigma + \varepsilon ||\boldsymbol{v}||^2_V. \tag{12}$$

For the second part of the proposition, we remark that the problem (Q_ε) is equivalent to the following problem for $\varepsilon > 0$: find $(\sigma_\varepsilon, \boldsymbol{u}_\varepsilon, p_\varepsilon) \in \Sigma_g \times V_g \times L^2(\Omega)$ such that

$$\begin{cases} \varepsilon \int_\Omega \left(\sigma_\varepsilon : \tau + (\mathrm{div}\sigma_\varepsilon) \cdot (\mathrm{div}\tau)\right) dx + \int_\Omega (\mathrm{div}\sigma_\varepsilon) \cdot (\mathrm{div}\tau)\, dx \\ \quad + \int_\Omega (\sigma_\varepsilon - 2\mu e(\boldsymbol{u}_\varepsilon) + p_\varepsilon I) : \tau\, dx = 0, \quad \forall \tau \in \Sigma_0, \\ \varepsilon \int_\Omega e(\boldsymbol{u}_\varepsilon) : e(\boldsymbol{v})\, dx + \int_\Omega (\mathrm{div}\boldsymbol{u}_\varepsilon)(\mathrm{div}\boldsymbol{v})\, dx \\ \quad - \int_\Omega (\sigma_\varepsilon - 2\mu e(\boldsymbol{u}_\varepsilon) + p_\varepsilon I) : 2\mu e(\boldsymbol{v})\, dx = 0, \quad \forall \boldsymbol{v} \in V_0, \\ \varepsilon \int_\Omega p_\varepsilon q\, dx + \int_\Omega (\sigma_\varepsilon - 2\mu e(\boldsymbol{u}_\varepsilon) + p_\varepsilon I) : q I\, dx = 0, \quad \forall q \in L^2(\Omega). \end{cases} \tag{13}$$

Let us choose $\tau = \sigma_\varepsilon - \sigma$, $\boldsymbol{v} = \boldsymbol{u}_\varepsilon - \boldsymbol{u}$ and $q = p_\varepsilon - p$ in the above weak formulation. We obtain

$$\begin{aligned} &||\sigma_\varepsilon - 2\mu e(\boldsymbol{u}_\varepsilon) + p_\varepsilon I||^2_{(L^2(\Omega))^{d\times d}} + ||\mathrm{div}\sigma_\varepsilon||^2_{(L^2(\Omega))^d} + ||\mathrm{div}\boldsymbol{u}_\varepsilon||^2_{L^2(\Omega)} \\ &\quad + \varepsilon(\sigma_\varepsilon, \sigma_\varepsilon - \sigma)_\Sigma + \varepsilon(\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon - \boldsymbol{u})_V + \varepsilon(p_\varepsilon, p_\varepsilon - p)_{L^2(\Omega)} = 0. \end{aligned} \tag{14}$$

From the above equation we first obtain that

$$||\sigma_\varepsilon||^2_\Sigma + ||\boldsymbol{u}_\varepsilon||^2_V + ||p_\varepsilon||^2_{L^2(\Omega)} \leq ||\sigma||^2_\Sigma + ||\boldsymbol{u}||^2_V + ||p||^2_{L^2(\Omega)},$$

that is the sequences $\boldsymbol{\sigma}_\varepsilon$, $\boldsymbol{u}_\varepsilon$ and p_ε are bounded. Therefore we can find subsequences which we still denote $\boldsymbol{\sigma}_\varepsilon$, $\boldsymbol{u}_\varepsilon$ and p_ε that weakly converge to $\boldsymbol{\theta} \in \Sigma_g$, $\boldsymbol{w} \in V_g$ and $q \in L^2(\Omega)$, respectively. Clearly we have $\boldsymbol{\sigma}_\varepsilon - 2\mu e(\boldsymbol{u}_\varepsilon) + p_\varepsilon I \rightharpoonup \boldsymbol{\theta} - 2\mu e(\boldsymbol{w}) + q I$ in $(L^2(\Omega))^{d\times d}$, $\mathrm{div}\boldsymbol{\sigma}_\varepsilon \rightharpoonup \mathrm{div}\boldsymbol{\theta}$ in $(L^2(\Omega))^d$ and $\mathrm{div}\boldsymbol{u}_\varepsilon \rightharpoonup \mathrm{div}\boldsymbol{w}$ in $L^2(\Omega)$. By subtracting $C(\boldsymbol{\sigma}, \boldsymbol{\sigma}_\varepsilon - \boldsymbol{\sigma})_\Sigma + C(\boldsymbol{u}, \boldsymbol{u}_\varepsilon - \boldsymbol{u})_V + C(p, p_\varepsilon - p)_{L^2(\Omega)}$ to equation (14), we obtain that

$$||\boldsymbol{\sigma}_\varepsilon - \boldsymbol{\sigma}||^2_\Sigma + ||\boldsymbol{u}_\varepsilon - \boldsymbol{u}||^2_V + ||p_\varepsilon - p||^2_{L^2(\Omega)} \leq ||\boldsymbol{\sigma}||^2_\Sigma + ||\boldsymbol{u}||^2_V + ||p||^2_{L^2(\Omega)},$$

and then

$$\begin{aligned} ||\boldsymbol{\sigma}_\varepsilon - 2\mu e(\boldsymbol{u}_\varepsilon) + p_\varepsilon I||^2_{(L^2(\Omega))^{d\times d}} + ||\mathrm{div}\boldsymbol{\sigma}_\varepsilon||^2_{(L^2(\Omega))^d} + ||\mathrm{div}\boldsymbol{u}_\varepsilon||^2_{L^2(\Omega)} \\ \leq C(||\boldsymbol{\sigma}||^2_\Sigma + ||\boldsymbol{u}||^2_V + ||p||^2_{L^2(\Omega)}), \end{aligned}$$

which implies that $\boldsymbol{\sigma}_\varepsilon - 2\mu e(\boldsymbol{u}_\varepsilon) + p_\varepsilon I \to 0$ in $(L^2(\Omega))^{d\times d}$, $\mathrm{div}\boldsymbol{\sigma}_\varepsilon \to \mathbf{0}$ in $(L^2(\Omega))^d$ and $\mathrm{div}\boldsymbol{u}_\varepsilon \to 0$ in $L^2(\Omega)^d$. It follows that $\boldsymbol{\theta} - 2\mu e(\boldsymbol{w}) + q I = 0$, $\mathrm{div}\boldsymbol{\theta} = \mathbf{0}$ and $\mathrm{div}\boldsymbol{w} = 0$ in Ω, that is $(\boldsymbol{\theta}, \boldsymbol{w}, q) \in \Sigma \times V \times L^2$

In the first mixed formulation, the variable $\boldsymbol{\lambda}_{\varepsilon,\gamma}$ plays the role of a Lagrange multiplier and has no particular physical meaning, contrary to $\boldsymbol{u}_{\varepsilon,\gamma}$, which approximates the velocity field. In the second mixed formulation, both variables $\boldsymbol{\sigma}_\varepsilon$ and $\boldsymbol{u}_\varepsilon$ have a physical meaning, since they approximate the stress and the velocity fields, respectively. Moreover, computing an approximated stress field $\boldsymbol{\sigma}_\varepsilon$ may be an advantage in some situations.

As a conclusion of theorem 3.2 and remark 4, in the presence of noisy data (which are possibly incompatible), the regularization parameter ε shall not be too small. A careful look at the second minimization problem show that it has the same form as the one solved in [26] for the Laplacian. So, with noisy data $(\boldsymbol{g}_0^\delta, \boldsymbol{g}_1^\delta)$ such that $||\boldsymbol{g}_0^\delta - \boldsymbol{g}_0||_{(H^{1/2}(\Gamma))^d} \leq \delta$ and $||\boldsymbol{g}_1^\delta - \boldsymbol{g}_1||_{(H^{-1/2}(\Gamma))^d} \leq \delta$, where $(\boldsymbol{g}_0, \boldsymbol{g}_1)$ is the exact data, we are then able, for this second mixed formulation, to select the regularization parameter ε as a function of the amplitude of noise δ. This is done by defining an auxiliary constrained minimization problem in $(\boldsymbol{\sigma}, \boldsymbol{v})$ in which we impose that $||\boldsymbol{v}|_\Gamma - \boldsymbol{g}_0^\delta||_{(H^{1/2}(\Gamma))^d} \leq \delta$ and $||\boldsymbol{\sigma} \cdot \boldsymbol{n}|_\Gamma - \boldsymbol{g}_1^\delta||_{(H^{-1/2}(\Gamma))^d} \leq \delta$ following the Morozov's discrepancy principle. In practice, instead of solving such constrained minimization problem considered as a primal problem, we solve its dual problem in the sense of [27], because such dual problem happens to be constraint free. The solution of the primal problem has then a simple expression in terms of the solution of the dual problem, and the most important remark is that the solution of the primal problem coincides with the quasi-reversibility solution for some special ε which is expressed as a function of δ and of the solution of the dual problem. This special ε corresponds to the Morozov's discrepancy principle in the quasi-reversibility method. Such strategy based on the Morozov's discrepancy principle and duality in optimization was shown to be very efficient in the Laplacian case (see [26]). Undoubtedly, the same method would also work in the case of the Stokes problem. On the contrary, we do not know yet how to use a similar method to select ε in the case of the first mixed formulation for some fixed parameter γ, for example $\gamma = 1$.

From the numerical point of view, the first mixed formulation is very simple since the unknown belongs to the space $(H^1(\Omega))^d \times (H^1(\Omega))^d$, which can be discretized with the help of the standard Lagrange finite elements (see the numerical section). This is in contrast with what we called a classical formulation of quasi-reversibility. The second mixed formulation is slightly more complicated since the unknown belongs to the space $\{\boldsymbol{\sigma} \in (L^2(\Omega))^{d \times d}, \quad \operatorname{div} \boldsymbol{\sigma} \in (L^2(\Omega))^d\} \times (H^1(\Omega))^d$, which for $\boldsymbol{\sigma}$ requires the use of Raviart-Thomas finite elements (see the numerical section). Note, however, that such finite elements are available in almost all finite element codes.

4. The level set method and the exterior approach

In this section we show how the level set method introduced in [13] in the case of the Laplace equation can be adapted to the Stokes system. The main difference between the standard level set methods and ours is that it relies on a simple Poisson equation instead of an eikonal equation such as described in [28]. Firstly, we define a sequence of open domains O_n which is based on the solution of problem (1) and which converges in a certain sense to the obstacle O. Secondly this fact enables us to design a level set technique to solve the inverse obstacle problem.

Assuming that the obstacle O and the velocity and pressure fields $(\boldsymbol{u}, p)$ solve problem (1), we consider a scalar field V such that

$$\left\{ \begin{array}{lll} V = |\boldsymbol{u}|_1 := \sum_{i=1}^d |u_i| & \text{in} & \Omega \\ V \in H^1_0(O) \quad \text{and} \quad V \leq 0 & \text{in} & O. \end{array} \right. \tag{15}$$

Since $\boldsymbol{u}|_{\partial O} = 0$, such field V satisfies $V \in H^1(\Omega)$ and it suffices to choose $V = 0$ in O to build some particular V in the whole domain Ω. Now, for $f \in H^{-1}(\Omega)$ such that (in the sense of $H^{-1}(\Omega)$)

$$f \geq \Delta V, \tag{16}$$

we consider the following sequence of open domains. For some open domain O_0 with $O \subset O_0 \Subset \Omega$, we define

$$O_{n+1} = O_n \setminus \text{supp}(\sup(\varphi_n, 0)), \tag{17}$$

where φ_n is the function in $H^1(O_n)$ such that $\psi_n := \varphi_n - V$ be the unique solution in $H^1_0(O_n)$ of the homogeneous Dirichlet problem

$$\Delta \psi_n = f - \Delta V \quad \text{in} \quad O_n. \tag{18}$$

Remark 5. In the definition of velocity field V the l_1 norm of vector $\boldsymbol{u}$ has been chosen. The l_2 or l_∞ norms could also have been chosen.

Remark 6. Because φ_n is only in $H^1(O_n)$, the definition (17) may seem a bit complicated. In the restricted case where φ_n is a continuous function in O_n up to the boundary, the definition of O_{n+1} given by (17) simply becomes

$$O_{n+1} = \{x \in O_n, \varphi_n(x) < 0\},$$

which means that the current obstacle is defined by the level set function φ_n, where n plays the role of a fictitious time.

Remark 7. In equation (18), the right-hand side $f - \Delta V$ is understood as the restriction to O_n of the distribution $f - \Delta V \in H^{-1}(\Omega)$. When O_n is a Lipschitz domain, then the traces of functions φ_n and $V = |\boldsymbol{u}|_1$ on ∂O_n are well defined and φ_n is the solution in $H^1(O_n)$ of the boundary value problem

$$\left\{ \begin{array}{lll} \Delta \varphi_n = f & \text{in} & O_n \\ \varphi_n = |\boldsymbol{u}|_1 & \text{on} & \partial O_n. \end{array} \right. \tag{19}$$

Such Poisson problem, which updates the level set function φ_n as a function of the solution $\boldsymbol{u}$, plays the role of the eikonal equation in standard level set techniques.

If we assume that the open domains O_n are uniformly Lipschitz with respect to n, we obtain the following theorem which guarantees convergence of the sequence of open domains O_n to the obstacle O in the sense of Hausdorff distance.

Theorem 4.1. *We consider the domains Ω, O, Γ, and the fields $(\boldsymbol{u}, p)$ as defined in section 2. Let $V \in H^1(\Omega)$ satisfy (15) and $f \in H^{-1}(\Omega)$ satisfy (16).*
Let O_0 denote an open domain such that $O \subset O_0 \Subset \Omega$, as well as the decreasing sequence of open domains O_n defined by (17) and (18).
With the additional assumption that the domains O_n are uniformly Lipschitz with respect to n (that is the O_n satisfy the cone property such that the angle of the cone be uniform with respect to n), we have

$$\overbrace{\bigcap_n O_n}^{\circ} = \mathcal{O},$$

with convergence in the sense of Hausdorff distance for open domains.

Proof. The proof is very close to that of theorem 2.5 in [13] for the Laplacian case, so we just give a sketch of it by insisting on what differs. Since the O_n form a decreasing sequence of open domains, from [29] (see section 2.2.3) the O_n converge in the sense of Hausdorff to the open set

$$\omega := \overbrace{\bigcap_n O_n}^{\circ}.$$

By using proposition 2 in [13] we first obtain given (16) that $O \subset O_n$ for all n and by using the fact that the inclusion is stable with respect to Hausdorff convergence (see section 2.2.3 in [29]) we obtain that $O \subset \omega$. The remainder of the proof consists in assuming $\omega \neq O$ and finding a contradiction.
From theorem 3.2.13 in [29], since the sequence of open domains O_n converge to ω in the sense of Hausdorff distance and since such open domains are uniformly Lipschitz with respect to n, ω is a Lipschitz continuous domain and the sequence of functions $_n \in H^1_0(O_n)$ given by (18) converge (in space $H^1_0(\;)$) to the solution $\in H^1_0(\omega)$ of the problem $\Delta \; = f - \Delta V$ in ω.
Since O is a Lipschitz continuous domain, that $\omega \neq O$ implies that the open domain $P := \omega \setminus \overline{O}$ is nonempty. Arguing exactly as in the proof of theorem 2.5 in [13], we obtain that $V \in H^1_0(P)$. For $i = 1, ..., d$, we have $|u_i| \leq V = |\boldsymbol{u}|_1$, then by using lemma 2.4 in [13] we obtain $|u_i| \in H^1_0(P)$, so that $\sup(u_i, 0), \sup(-u_i, 0) \in H^1_0(P)$, and lastly $u_i = \sup(u_i, 0) - \sup(-u_i, 0) \in H^1_0(P)$, that is $\boldsymbol{u} \in (H^1_0(P))^d$.
By arguing as in the proof of lemma 1 (it suffices to replace $\tilde{O} \setminus \overline{O^2}$ by P), it follows that $\boldsymbol{u} = \boldsymbol{0}$ in P, that is $\boldsymbol{u} = \boldsymbol{0}$ in Ω from unique continuation, which contradicts the fact that $\boldsymbol{g_0} \neq \boldsymbol{0}$ and completes the proof. □

In view of equation (19), the definition of the open domains O_n uniquely depends on the values of the velocity field $\boldsymbol{u}$ outside the obstacle and on f. The field $\boldsymbol{u}$ is unknown , while from (16) f shall be larger than an unknown function. This inspires us the exterior approach to solve the inverse obstacle problem (1). The idea is to replace the exact solution $\boldsymbol{u}$ by an approximated one obtained from the Cauchy data $(\boldsymbol{g_0}, \boldsymbol{g_1})$, for example by solving a quasi-reversibility problem, and to choose for f a sufficiently large function chosen arbitrarily. We hence obtain the following algorithm.

Algorithm :

1. Choose an initial guess O_0 such that $O \subset O_0 \Subset$.
2. First step: the domain O_n being given, solve one of the mixed quasi-reversibility problem $(\;_{\varepsilon,\gamma})$ given by (3) or (Q_ε) given by (11) in the domain $\Omega_n := \;\setminus \overline{O_n}$, from data $(\boldsymbol{g_0}, \boldsymbol{g_1})$ on Γ, for some selected parameters ε, γ. The solution in terms of the velocity field is denoted $\boldsymbol{u}_n$.
3. Second step: the function $\boldsymbol{u}_n$ being given in Ω_n, solve the boundary value problem
$$\begin{cases} \Delta \phi_n &= & f & \text{in} & O_n \\ \phi_n &= & |\boldsymbol{u}_n|_1 & \text{in} & \partial O_n \end{cases}$$
for some selected f and compute
$$O_{n+1} = \{x \in O_n, \phi_n(x) < 0\}.$$
4. Go back to the first step until the stopping criteria is reached.

Remark 8. It seems hard to analyze the convergence of the above global algorithm on the one hand with respect to the regularization parameters ε, γ and on the other hand with respect to the number n of iterations. In fact, for exact data $(\boldsymbol{g}_0, \boldsymbol{g}_1)$ the above theorems are partial convergence results in the sense that theorems 3.1, 3.3 prove convergence of the computed velocity field to the exact one with respect to (ε, γ) for a known obstacle, while on the contrary theorem 4.1 proves convergence of the computed obstacle to the exact one with respect to n for known velocity field.

5. Discretization of quasi-reversibility

Both mixed formulations of quasi-reversibility can be discretized with the help of classical conforming finite elements. More precisely, we need to approach spaces $V = (H^1(\Omega))^d$ and $\Sigma = \{\sigma \in (L^2(\Omega))^{d\times d},\ \mathrm{div}\sigma \in (L^2(\Omega))^d\}$ for $d = 2, 3$, which is a classical problem in the field of fluid mechanics or mechanical engineering. We restrict ourselves to polygonal domains ($d = 2$)/polyhedral domains ($d = 3$) and consider a family of triangulations $\mathcal{T}_h$ of domain $\overline{\Omega}$ such that the diameter of each triangle/tetrahedron $K \in \mathcal{T}_h$ is bounded by $h > 0$ and such that $\mathcal{T}_h$ is regular in the sense of [21]. We assume that $\overline{\Gamma}$ is formed by the union of edges/faces of some triangles/tetrahedra of $\mathcal{T}_h$. For each triangle/tetrahedron $K \in \mathcal{T}_h$ and $k \in \mathbb{N}$, $\mathbb{P}_k(K)$ denotes the space of polynomial functions of degree lower or equal to $k \in \mathbb{N}$. We introduce two different finite elements : the standard Lagrange finite element P_h^k for $k \geq 1$ and the Raviart-Thomas finite element RT_h^k for $k \in \mathbb{N}$. They are defined in [21] and [30], respectively. They satisfy in particular for $k \geq 1$,

$$P_h^k \subset \{v_h \in H^1(\Omega),\quad v_h|_K \in \mathbb{P}_k(K),\ \forall K \in \mathcal{T}_h\},$$

while for $k \in \mathbb{N}$,

$$RT_h^k \subset \{\boldsymbol{q}_h \in (L^2(\Omega))^d,\ \mathrm{div}\boldsymbol{q}_h \in L^2(\Omega),\quad \boldsymbol{q}_h \in (\mathbb{P}_k(K))^d + \boldsymbol{x}\,\mathbb{P}_k(K),\ \forall K \in \mathcal{T}_h\},$$

where $\boldsymbol{x} \in \mathbb{R}^d$ is the spatial coordinate. It is important to note that for $k \geq 1$ the trace of some $v_h \in P_h^k$ is continuous across the edge/face between too elements, while for some $\boldsymbol{q}_h \in RT_h^k$, the trace of $\boldsymbol{q}_h \cdot \boldsymbol{n}$ is continuous across such edge/face, where $\boldsymbol{n}$ is the normal to such boundary.

Let us first introduce a discretized version of the first mixed formulation (3) based on the Cauchy data $(\boldsymbol{g}_0, \boldsymbol{g}_1) \in (H^{\frac{1}{2}}(\Gamma))^d \times (H^{-\frac{1}{2}}(\Gamma))^d$ on Γ. We consider the restricted case $\boldsymbol{g}_0 \in (H^{\frac{1}{2}}(\Gamma))^d \cap (C^0(\overline{\Gamma}))^d$, and we define $\boldsymbol{g}_{0\,h}$ as its interpolant over the space of traces on Γ of $(P_h^k)^d$-functions for some given $k \geq 1$, that is $\boldsymbol{g}_{0\,h}$ has the same degrees of freedom on Γ as $\boldsymbol{g}_0$ (since such degrees of freedom are punctual values of the function, $\boldsymbol{g}_0$ needs to be continuous on Γ). Introducing for $k \geq 1$ the sets

$$V_{gh}^k = \{\boldsymbol{u}_h \in (P_h^k)^d,\quad \boldsymbol{u}_h|_\Gamma = \boldsymbol{g}_{0\,h}\},$$

$$V_{0\,h}^k = \{\boldsymbol{u}_h \in (P_h^k)^d,\quad \boldsymbol{u}_h|_\Gamma = \boldsymbol{0}\},\quad \tilde{V}_{0\,h}^k = \{\boldsymbol{\lambda}_h \in (P_h^k)^d,\quad \boldsymbol{\lambda}_h|_{\tilde{\Gamma}} = \boldsymbol{0}\},$$

the discretized formulation $(P^k_{\varepsilon,\gamma,h})$ is for $\varepsilon, \gamma, \mu > 0$ and $k \geq 1$: find $(\boldsymbol{u}_{\varepsilon,\gamma,h}, \boldsymbol{\lambda}_{\varepsilon,\gamma,h})$ in $V^k_{gh} \times \tilde{V}^k_{0h}$ such that

$$(20)\qquad \begin{cases} 2\mu\varepsilon \displaystyle\int_\Omega e(\boldsymbol{u}_{\varepsilon,\gamma,h}) : e(\boldsymbol{v}_h)\, d\boldsymbol{x} + \int_\Omega \operatorname{div}\boldsymbol{u}_{\varepsilon,\gamma,h} \operatorname{div}\boldsymbol{v}_h\, d\boldsymbol{x} \\ \qquad + 2\mu \displaystyle\int_\Omega e(\boldsymbol{v}_h) : e(\boldsymbol{\lambda}_{\varepsilon,\gamma,h})\, d\boldsymbol{x} = 0, \quad \forall \boldsymbol{v}_h \in V^k_{0h}, \\ 2\mu \displaystyle\int_\Omega e(\boldsymbol{u}_{\varepsilon,\gamma,h}) : e(\boldsymbol{\mu}_h)\, d\boldsymbol{x} - \frac{1}{\varepsilon}\int_\Omega \operatorname{div}\boldsymbol{\lambda}_{\varepsilon,\gamma,h} \operatorname{div}\boldsymbol{\mu}_h\, d\boldsymbol{x} \\ -\gamma \displaystyle\int_\Omega e(\boldsymbol{\lambda}_{\varepsilon,\gamma,h}) : e(\boldsymbol{\mu}_h)\, d\boldsymbol{x} = \int_{\Gamma} \boldsymbol{g_1} \cdot \boldsymbol{\mu}_h\, ds, \quad \forall \boldsymbol{\mu}_h \in \tilde{V}^k_{0h}. \end{cases}$$

We have the following estimate of the discrepancy between the solution of problem $(P^k_{\varepsilon,\gamma,h})$ given by (20) and the solution of problem $(P_{\varepsilon,\gamma})$ given by (3).

Theorem 5.1.

Remark 9. We can remark that we can improve the constant $C(\varepsilon,\gamma)$ by using the following well-posed discretized formulation of (4) involving the pressure field: for $\varepsilon,\gamma,h>0$ and $k\geq 1$: find $(\boldsymbol{u}_{\varepsilon,\gamma,h},\boldsymbol{\lambda}_{\varepsilon,\gamma,h},p_{\varepsilon,\gamma,h})$ in $V^k_{gh}\times\tilde{V}^k_{0h}\times P^{k-1}_h$ such that

$$(21)\qquad\left\{\begin{array}{l} 2\varepsilon\mu\displaystyle\int_\Omega e(\boldsymbol{u}_{\varepsilon,\gamma,h}):e(\boldsymbol{v}_h)\,dx+\int_\Omega \mathrm{div}\boldsymbol{u}_{\varepsilon,\gamma,h}\,\mathrm{div}\boldsymbol{v}_h\,dx\\ \quad+2\mu\displaystyle\int_\Omega e(\boldsymbol{v}_h):e(\boldsymbol{\lambda}_{\varepsilon,\gamma,h})\,dx=0,\quad\forall\boldsymbol{v}_h\in V^k_{0h},\\ \quad 2\mu\displaystyle\int_\Omega e(\boldsymbol{u}_{\varepsilon,\gamma,h}):e(\boldsymbol{\mu}_h)\,dx-\int_\Omega p_{\varepsilon,\gamma,h}\,\mathrm{div}\boldsymbol{\mu}_h\,dx\\ -\gamma\displaystyle\int_\Omega e(\boldsymbol{\lambda}_{\varepsilon,\gamma,h}):e(\boldsymbol{\mu}_h)\,dx=\int \boldsymbol{g_1}\cdot\boldsymbol{\mu}_h\,ds,\quad\forall\boldsymbol{\mu}_h\in\tilde{V}^k_{0h},\\ \varepsilon\displaystyle\int_\Omega p_{\varepsilon,\gamma,h}q_h\,dx-\int_\Omega q_h\,\mathrm{div}\boldsymbol{\lambda}_{\varepsilon,\gamma,h}\,dx=0,\quad\forall q_h\in P^{k-1}_h,\end{array}\right.$$

where

$$P^k_h=\{p_h\in L^2(\Omega),\quad p_h|_K\in P_k(K),\ \forall K\in\mathcal{T}_h\}.$$

If $\varepsilon\gamma\leq 1$ and $(\boldsymbol{u}_{\varepsilon,\gamma},\boldsymbol{\lambda}_{\varepsilon,\gamma},p_{\varepsilon,\gamma})$ belongs to $(H^{k+1}(\Omega))^d\times(H^{k+1}(\Omega))^d\times H^k(\Omega)$, then

$$||\boldsymbol{u}_{\varepsilon,\gamma,h}-\boldsymbol{u}_{\varepsilon,\gamma}||_V+||\boldsymbol{\lambda}_{\varepsilon,\gamma,h}-\boldsymbol{\lambda}_{\varepsilon,\gamma}||_V+||p_{\varepsilon,\gamma,h}-p_{\varepsilon,\gamma}||_{L^2(\Omega)}$$

$$\leq C'(\varepsilon,\gamma)\,h^k(||\boldsymbol{u}_{\varepsilon,\gamma}||_{(H^{k+1}(\Omega))^d}+||\boldsymbol{\lambda}_{\varepsilon,\gamma}||_{(H^{k+1}(\Omega))^d}+||p_{\varepsilon,\gamma}||_{H^k(\Omega)}),$$

where

$$C'(\varepsilon,\gamma)=\frac{c}{\min(2\mu\varepsilon,\varepsilon\gamma)},$$

and $c>0$ is independent of ε,γ,h. The drawback of formulation (21) compared to (20) is that we compute the additional unknown $p_{\varepsilon,\gamma,h}$.

Let us introduce a discretized version of the second mixed formulation (11). In the following we consider the spaces $(PT^k_h)^d\subset\Sigma$ such that $\sigma_h\in(PT^k_h)^d$ if $\sigma_{ih}\in PT^k_h$ for all $i=1,...,d$, where σ_{ih} denotes the $i-$ line of the $d\times d$ matrix σ_h. We consider the restricted case $\boldsymbol{g_0}\in(H^{\frac12}(\Gamma))^d\cap(C^0(\overline{\Gamma}))^d$ and $\boldsymbol{g_1}\in(L^2(\Gamma))^d$. In addition to $\boldsymbol{g_0}_h$ we define $\boldsymbol{g_{1h}}$ as the interpolant of $\boldsymbol{g_1}$ over the space of the normal stresses on Γ of stress fields in $(PT^k_h)^d$ (since the degrees of freedom of such normal stresses are integrals over Γ, $\boldsymbol{g_1}$ needs to be integrable over Γ). Introducing the sets

$$\Sigma^k_{gh}=\{\sigma_h\in(PT^k_h)^d,\quad\sigma_h\cdot\boldsymbol{n}|_\Gamma=\boldsymbol{g_{1h}}\},\quad\Sigma^k_{0h}=\{\sigma_h\in(PT^k_h)^d,\ \sigma_h\cdot\boldsymbol{n}|_\Gamma=\boldsymbol{0}\},$$

the discretized formulation $(Q^k_{\varepsilon,h})$ is for $\varepsilon,h>0$ and $k\geq 1$: find $(\sigma_{\varepsilon,h},\boldsymbol{u}_{\varepsilon,h})\in\Sigma^{k-1}_{gh}\times V^k_{gh}$ such that

(22)

$$\left\{\begin{array}{l}\varepsilon\displaystyle\int_\Omega(\sigma_{\varepsilon,h}:\tau_h+(\mathrm{div}\sigma_{\varepsilon,h})\cdot(\mathrm{div}\tau_h))\,dx+\int_\Omega(\mathrm{div}\sigma_{\varepsilon,h})\cdot(\mathrm{div}\tau_h)\,dx\\ +\displaystyle\int_\Omega\left(\sigma_{\varepsilon,h}-2\mu e(\boldsymbol{u}_{\varepsilon,h})-\frac{1}{d+\varepsilon}\mathrm{tr}(\sigma_{\varepsilon,h}-2\mu e(\boldsymbol{u}_{\varepsilon,h}))\,I\right):\tau_h\,dx=0,\quad\forall\tau_h\in\Sigma^{k-1}_{0h},\\ \varepsilon\displaystyle\int_\Omega e(\boldsymbol{u}_{\varepsilon,h}):e(\boldsymbol{v}_h)\,dx+\int_\Omega(\mathrm{div}\boldsymbol{u}_{\varepsilon,h})(\mathrm{div}\boldsymbol{v}_h)\,dx\\ -\displaystyle\int_\Omega\left(\sigma_{\varepsilon,h}-2\mu e(\boldsymbol{u}_{\varepsilon,h})-\frac{1}{d+\varepsilon}\mathrm{tr}(\sigma_{\varepsilon,h}-2\mu e(\boldsymbol{u}_{\varepsilon,h}))\,I\right):2\mu e(\boldsymbol{v}_h)\,dx=0,\quad\forall\boldsymbol{v}_h\in V^k_{0h}.\end{array}\right.$$

We have the following estimate of the discrepancy between the solution of problem $(Q^k_{\varepsilon,h})$ given by (22) and the solution of problem (Q_ε) given by (11).

Theorem 5.2. *For all $\gamma, h > 0$ and $k \geq 1$, the problem (22) has a unique solution $(\sigma_{\varepsilon,h}, \boldsymbol{u}_{\varepsilon,h})$ in $\Sigma_{gh}^{k-1} \times V_{gh}^k$. Furthermore, if $\gamma \leq 1$ and $(\sigma_\varepsilon, \boldsymbol{u}_\varepsilon)$ belongs to $\{\sigma \in (H^k(\Omega))^{d\times d}, \operatorname{div}\sigma \in (H^k(\Omega))^d\} \times (H^{k+1}(\Omega))^d$, then*

$$||\sigma_{\varepsilon,h} - \sigma_\varepsilon||_\Sigma + ||\boldsymbol{u}_{\varepsilon,h} - \boldsymbol{u}_\varepsilon||_V \leq$$
$$\frac{c}{\sqrt{\gamma}} h^k \left(||\sigma_\varepsilon||_{(H^k(\Omega))^{d\times d}} + ||\operatorname{div}\sigma_\varepsilon||_{(H^k(\Omega))^d} + ||\boldsymbol{u}_\varepsilon||_{(H^{k+1}(\Omega))^d}\right),$$

where $c > 0$ is independent of γ, h.

Proof. Well-posedness of problem (22) is based on the same arguments as in the proof of theorem 3.3. By using the same notations as in the proof of theorem 3.3, we define $X_\varepsilon = (\sigma_\varepsilon, \boldsymbol{u}_\varepsilon)$, $X = (\Sigma, \boldsymbol{U})$ and $\hat{X}_\varepsilon = X_\varepsilon - X$. Considering the interpolant $\Sigma_h \in (RT_h^{k-1})^d$ of Σ and the interpolant $\boldsymbol{U}_h \in (P_h^k)^d$ of $\boldsymbol{U}$, we also define $X_{\varepsilon,h} = (\sigma_{\varepsilon,h}, \boldsymbol{u}_{\varepsilon,h})$, $X_h = (\Sigma_h, \boldsymbol{U}_h)$ and $\hat{X}_{\varepsilon,h} = X_{\varepsilon,h} - X_h$. For all $Z_h = (\tau_h, \boldsymbol{v}_h) \in \Sigma_{0\,h}^{k-1} \times V_{0\,h}^k \subset \Sigma_0 \times V_0$, we have from (11) and (22),

$$B_\varepsilon(X_\varepsilon; Z_h) = 0 = B_\varepsilon(X_{\varepsilon,h}; Z_h),$$

that is

$$B_\varepsilon(X_\varepsilon - X_{\varepsilon,h}; Z_h) = 0.$$

Since the bilinear form B_ε is symmetric, we conclude that $X_{\varepsilon,h}$ minimizes the functional $B_\varepsilon(X_\varepsilon - W_h; X_\varepsilon - W_h)$ over all $W_h \in \Sigma_{gh}^{k-1} \times V_{gh}^k$. With the help of (12) and the fact that $\gamma \leq 1$, we obtain

$$||X_\varepsilon - X_{\varepsilon,h}|| \leq \frac{c}{\sqrt{\gamma}} \inf_{W_h \in \Sigma_{gh}^{k-1} \times V_{gh}^k} ||X_\varepsilon - W_h||,$$

where $||\cdot||$ is the product norm of $\Sigma \times V$. The estimate follows from the classical interpolation results in spaces Σ and V (see [30]). In particular, if $\sigma_\varepsilon \in (H^k(\Omega))^{d\times d}$ is such that $\operatorname{div}\sigma_\varepsilon \in (H^k(\Omega))^d$ and $\tilde{\sigma}_{\varepsilon,h}$ is its interpolant in $(RT_h^{k-1})^d$, we have

$$||\sigma_\varepsilon - \tilde{\sigma}_{\varepsilon,h}||_\Sigma \leq c\, h^k \left(||\sigma_\varepsilon||_{(H^k(\Omega))^{d\times d}} + ||\operatorname{div}\sigma_\varepsilon||_{(H^k(\Omega))^d}\right)$$

where $c > 0$ is independent of γ, h. □

6. Some numerical experiments

6.1. The ill-posed Cauchy problem. In this first numerical section, we test our two discretized mixed formulations (20) and (22) in order to solve the ill-posed Cauchy problem (2) in the 2D configuration proposed in [6]. The domain Ω is the region contained between two concentric circles centered at $(0,0)$ and of radii 1 and 2, respectively. The support Γ of the data $(\boldsymbol{g}_0, \boldsymbol{g}_1)$ is the outer circle, no data being given on the inner circle. We try to recover the rather "singular" solution proposed in [6], which is given analytically by

$$\boldsymbol{u} = \frac{1}{4\pi}\left(\log\frac{1}{\sqrt{(x_1-0.5)^2+x_2^2}} + \frac{(x_1-0.5)^2}{(x_1-0.5)^2+x_2^2}, \frac{x_2(x_1-0.5)}{(x_1-0.5)^2+x_2^2}\right),$$
$$p = \frac{1}{2\pi}\frac{x_1-0.5}{(x_1-0.5)^2+x_2^2}. \tag{23}$$

The two formulations (20) and (22) are applied in a triangular mesh which is obtained by dividing the outer boundary by 100 and the inner boundary by 50, a regular mesh being generated by the Freefem software [31] such that $h \simeq 0.5$. The finite element computations are also performed with [31]. In the following figures

presenting the results of the two mixed quasi-reversibility formulations, the continuous red line represents the exact solution, the blue crosses represent the approximate solution obtained with formulation (20), while the green dots represent the approximate solution obtained with formulation (22).
First, we test free of noise data, with parameters $\varepsilon = 10^{-4}$ and $\gamma = 1$. The figures below show the different components of the approximated velocity field $\boldsymbol{u}$ and stress tensor σ that are retrieved on the inner circle. Note that the approximate stress tensor σ is a direct output of the mixed formulation (22), while it is obtained after applying (20) by using the relationships

$$p_{\varepsilon,\gamma,h} = \mathrm{div}\boldsymbol{\lambda}_{\varepsilon,\gamma,h}, \quad \sigma_{\varepsilon,\gamma,h} = 2\mu e(\boldsymbol{u}_{\varepsilon,\gamma,h}) - p_{\varepsilon,\gamma,h} I.$$

We study in particular the influence of the order of discretization, which is characterized by k. In figure 1 we have $k = 1$, while in figure 2 we have $k = 2$. The results of figures 1 and 2 show that the quality of the reconstruction is much better for $k = 2$ than for $k = 1$, independently of the formulation we use, which was expected from theorems 5.1 and 5.2. The better choice $k = 2$ is made in the remainder of the paper. Broadly speaking, the figure 2 show that for $k = 2$ with exact data the two mixed formulations (20) and (22) provide approximately the same results.
We now introduce some noisy data in order to test the robustness of the two methods. Precisely, we impose a pointwise random noise to the Dirichlet data $\boldsymbol{g_0}$, the Neumann data $\boldsymbol{g_1}$ remaining free of noise. The relative error in L^∞ norm for such noisy Dirichlet data is denoted by δ_r. We test three different amplitudes of noise, namely $\delta_r = 0.01$, $\delta_r = 0.05$ and $\delta_r = 0.1$, with a larger value of ε as before, namely $\varepsilon = 10^{-3}$, keeping $\gamma = 1$. In order to improve the results, we have applied a regularization process to the noisy data. It consists, for $\boldsymbol{g_0}^\delta \in L^2(\Gamma)$ such that $||\boldsymbol{g_0}^\delta - \boldsymbol{g_0}||_{L^2(\Gamma)} \leq \delta$, to find the unique function $\tilde{\boldsymbol{g}}_0^\delta \in H^1(\Gamma)$ that minimizes $||\tilde{\boldsymbol{g}}_0||_{H^1(\Gamma)}$ under the constraint $||\tilde{\boldsymbol{g}}_0 - \boldsymbol{g_0}^\delta||_{L^2(\Gamma)} \leq \delta$. In the figures 3, 4 and 5, the exact data $\boldsymbol{g_0} = (u_1, u_2)$ on the outer circle are represented by the red continuous line, the noisy data $\boldsymbol{g_0}^\delta$ by the green dots, and the smoothed data $\tilde{\boldsymbol{g}}_0^\delta$ by the black dashed line, for $\delta_r = 0.01$, $\delta_r = 0.05$ and $\delta_r = 0.1$, respectively. The reconstructions obtained on the inner circle for $\delta_r = 0.01$, $\delta_r = 0.05$ and $\delta_r = 0.1$ by using our smoothing procedure for noisy data and each of the two mixed formulations are given on figure 6, 7 and figure 8, respectively. We can see that in the case of large amplitude of noise, the reconstructions of the stress tensor are much better with the formulation (22) than with the formulation (20), which was expected since the discretization of the stress tensor is more precise in formulation (22) than in formulation (20).

6.2. The inverse obstacle problem. In this second numerical section we test the resolution of the inverse obstacle problem following the algorithm of section 4. In order to obtain a reasonable identification we restrict to a small amplitude of noise $\delta_r = 0.01$. In this case, we have seen that the accuracy of the two mixed formulations are more or less the same. Furthermore, since the obstacle is of Dirichlet type, only the velocity field needs to be correctly approximated. For these reasons, we have chosen to use the first mixed formulation (20), which is slightly more simple. The domain Ω is the unit disk centered at $(0,0)$ and the support of data Γ is the whole unit circle $\partial\Omega$. The Dirichlet obstacle O has a variable shape which will be specified later on. In order to obtain artificial data $(\boldsymbol{g_0}, \boldsymbol{g_1})$ on Γ, we compute a forward problem for the Stokes system with the help of a standard mixed finite

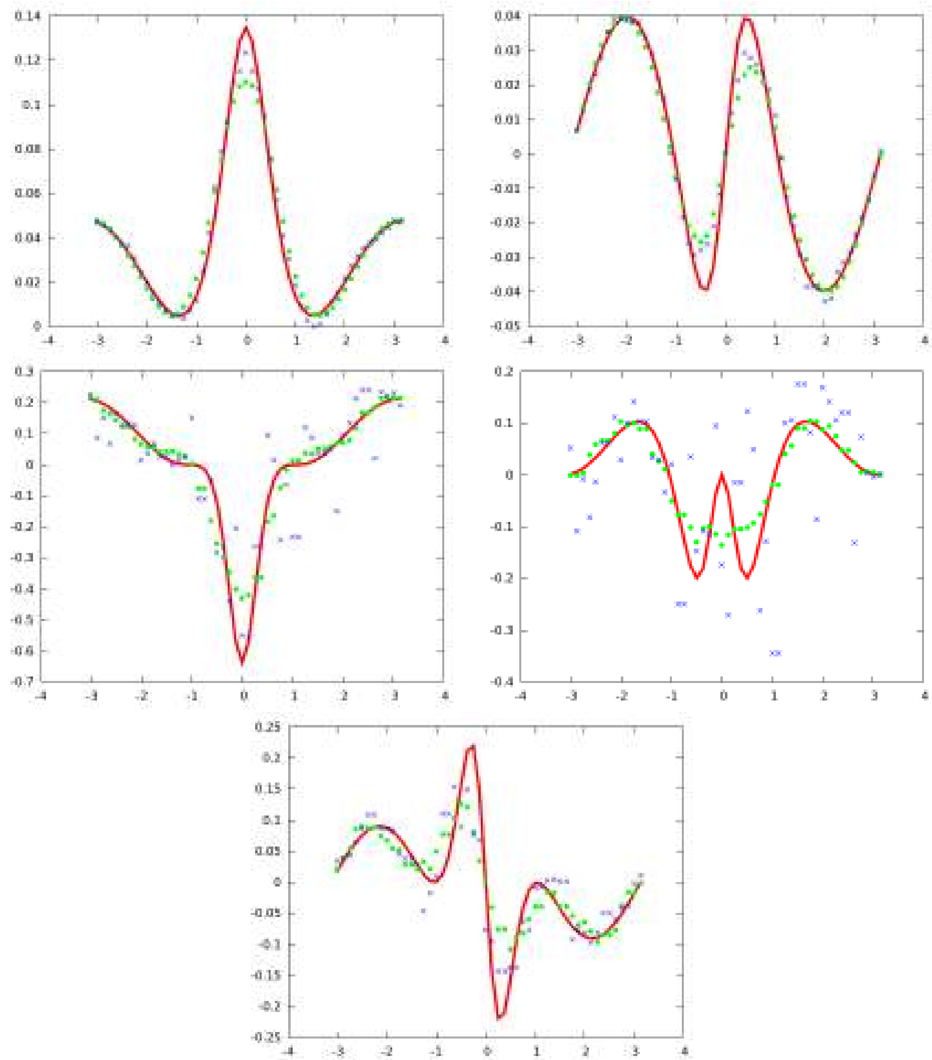

Figure 1. Case of = 1. Top left: u_1, Top right: u_2, Middle left: σ_{11}, Middle right: σ_{22}, Bottom: σ_{12}

element formulation [32] based on the nonhomogeneous Dirichlet boundary condition $\boldsymbol{u} = \boldsymbol{u}_0$ on ∂ , $\boldsymbol{u}_0$ being a constant vector, and of course $\boldsymbol{u} = 0$ on ∂O. We hence simulate the case of a fixed obstacle immersed in a flow the velocity of which is fixed to a constant $\boldsymbol{u}_0$ far away from such obstacle.

The inverse algorithm is applied by using a mesh which is generated by dividing the unit circle ∂ by 200 and which is different from the one used to obtain the artificial data with he help of the forward computation. Concerning the mixed formulation (20), we have taken $= 2$, $\varepsilon = 10^{-3}$ and $\gamma = 1$ as before. Concerning the discretized Poisson equation associated with (19), the obstacle O_n is approximated by a polygon, that is its boundary coincides with some edges of our triangular mesh, in particular the initial guess O_0 is a polygonal approximation of the circle centered at $(0, 0)$ and of radius 0.8. We have also used a standard finite element ^k_h formulation

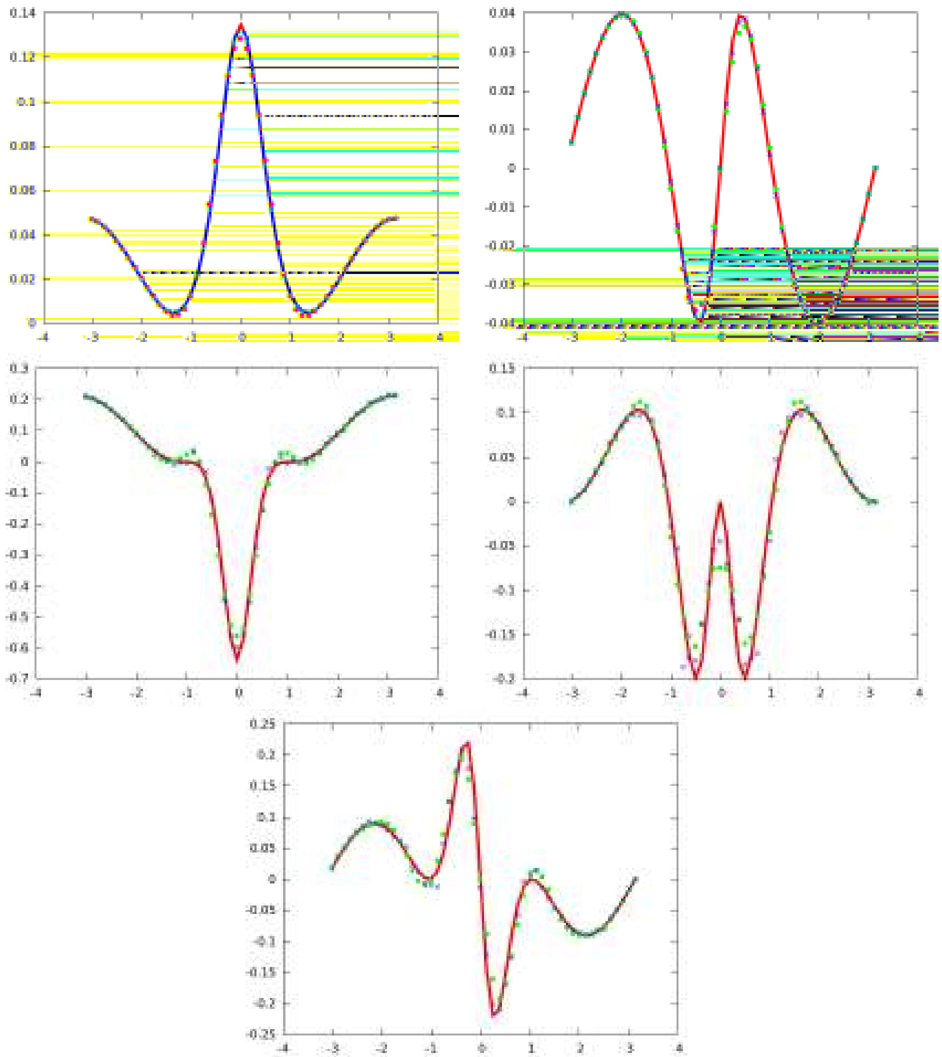

FIGURE 2. Case of = 2. Top left: u_1, Top right: u_2, Middle left: σ_{11}, Middle right: σ_{22}, Bottom: σ_{12}

with = 2 to solve (19) in the polygons that approximate O_n, the right-hand side f being chosen as a large constant which may be different from a case to another and which will be specified hereafter. Concerning the stopping criterion, we use the same as the one described in [13]. It is important to have in mind that a single mesh is used to solve the inverse problem: both the quasi-reversibility method and the level set method are based on such mesh. In the figure 9, we present the identification results for two different obstacles, as well as the iterated domains O_n until convergence for the first obstacle. In the figure 10, we present the identification results for two circles, with two sets of data, namely one produced by the velocity $\boldsymbol{u}_0 = (1, -1)\ \sqrt{2}$ on ∂ and the other one produced by the velocity $\boldsymbol{u}_0 = (1, 1)\ \sqrt{2}$ on ∂. The second set of data is more tricky because the two circles are in this case oriented in the direction of the flow, which implies that one circle is to some extent

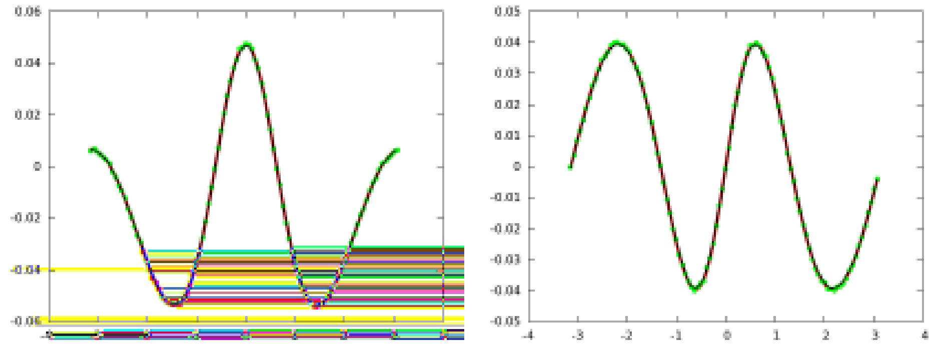

Figure 3. Amplitude of noise $\delta_r = 0.01$. Left: u_1, Right: u_2

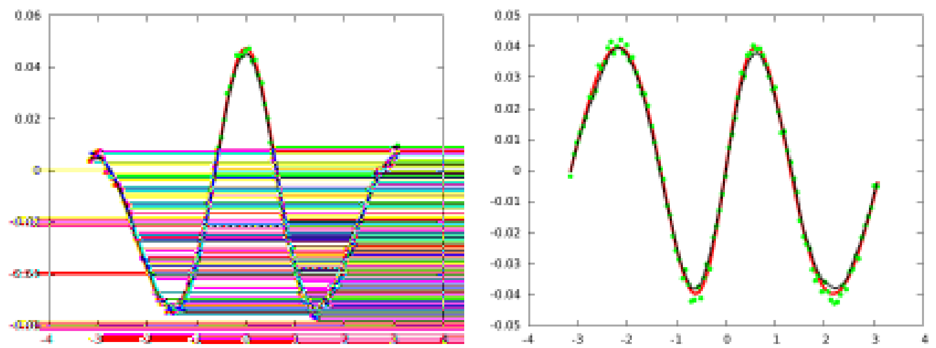

Figure 4. Amplitude of noise $\delta_r = 0.05$. Left: u_1, Right: u_2

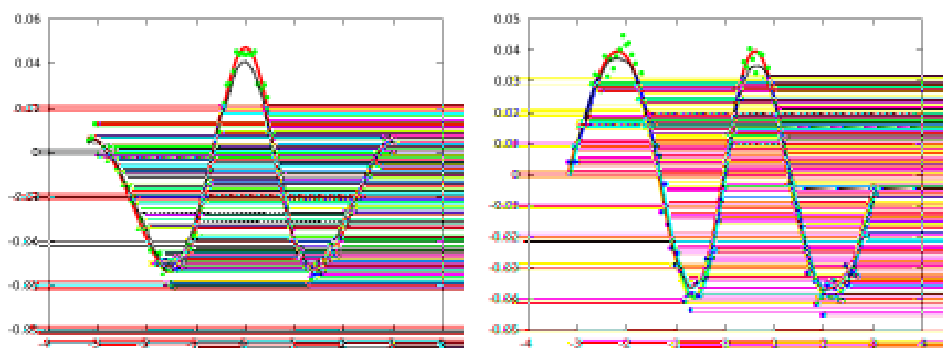

Figure 5. Amplitude of noise $\delta_r = 0.1$. Left: u_1, Right: u_2

hidden by the other. In the first case, the iterated domains O_n until convergence are represented in the figure. We complete this numerical section by the identification results of one circle given in figure 11 for two relative amplitudes of noise, 0.01 and 0.05 respectively, in order to show the impact of noise, keeping $\varepsilon = 10^{-3}$ and $\gamma = 1$. Once again we recall that for a large amplitude of noise, it would be preferable to use the second mixed formulation (22) and the procedure described in section

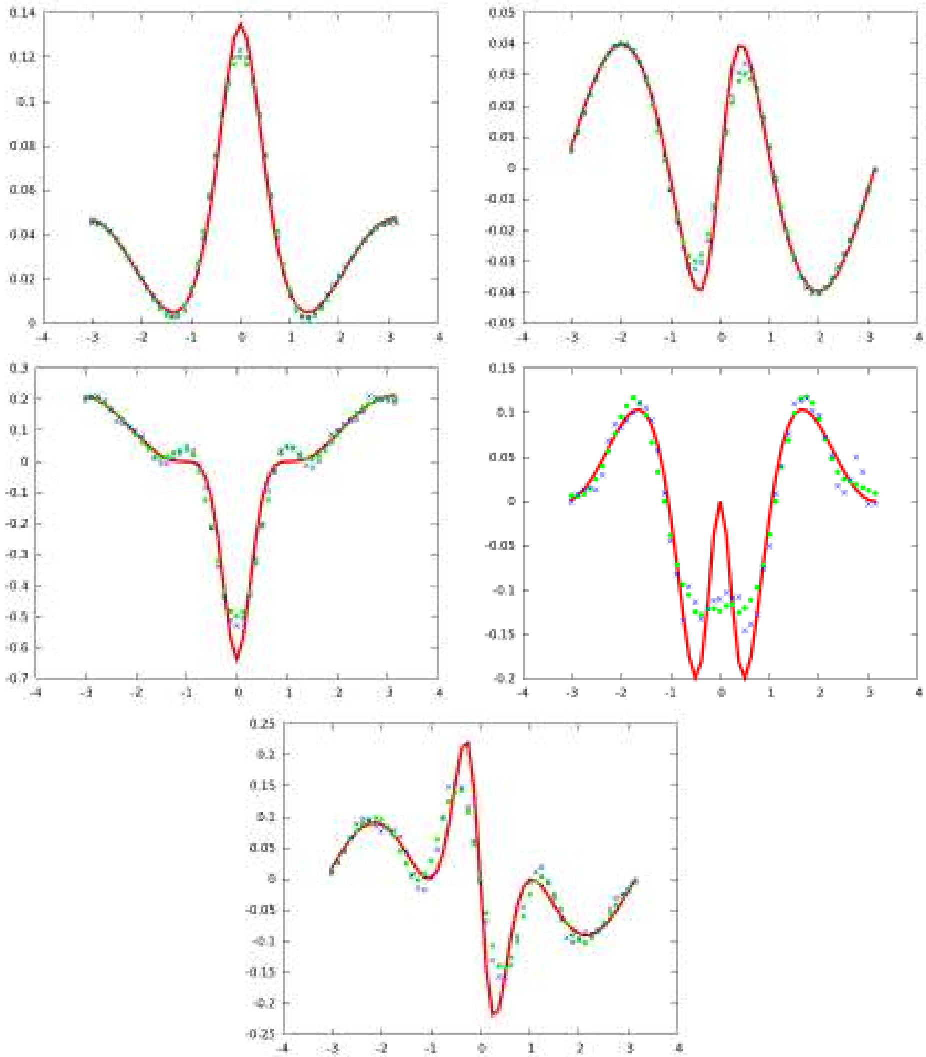

FIGURE 6. Amplitude of noise $\delta_r = 0.01$. Top left: u_1, Top right: u_2, Middle left: σ_{11}, Middle right: σ_{22}, Bottom: σ_{12}

3.4 to select the regularization parameter ε. Such procedure would represent an

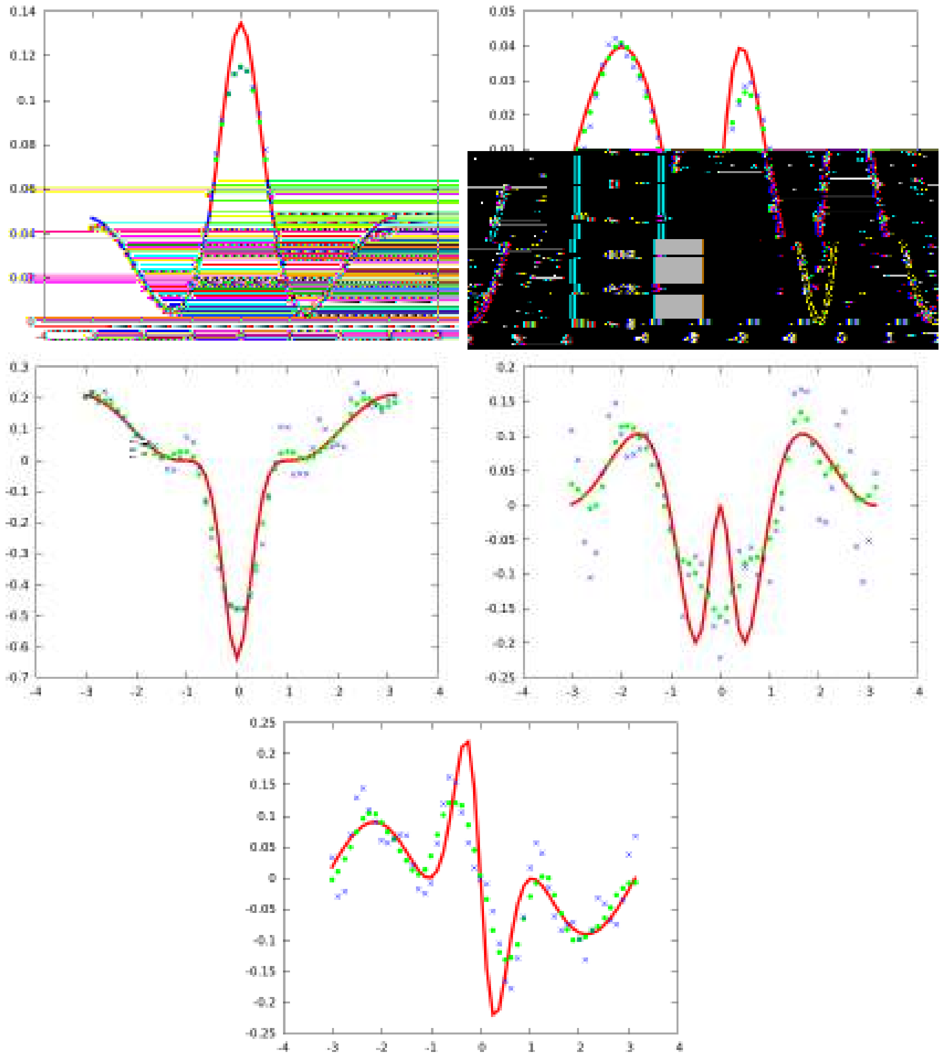

Figure 7. Amplitude of noise $\delta_r = 0.05$. Top left: u_1, Top right: u_2, Middle left: σ_{11}, Middle right: σ_{22}, Bottom: σ_{12}

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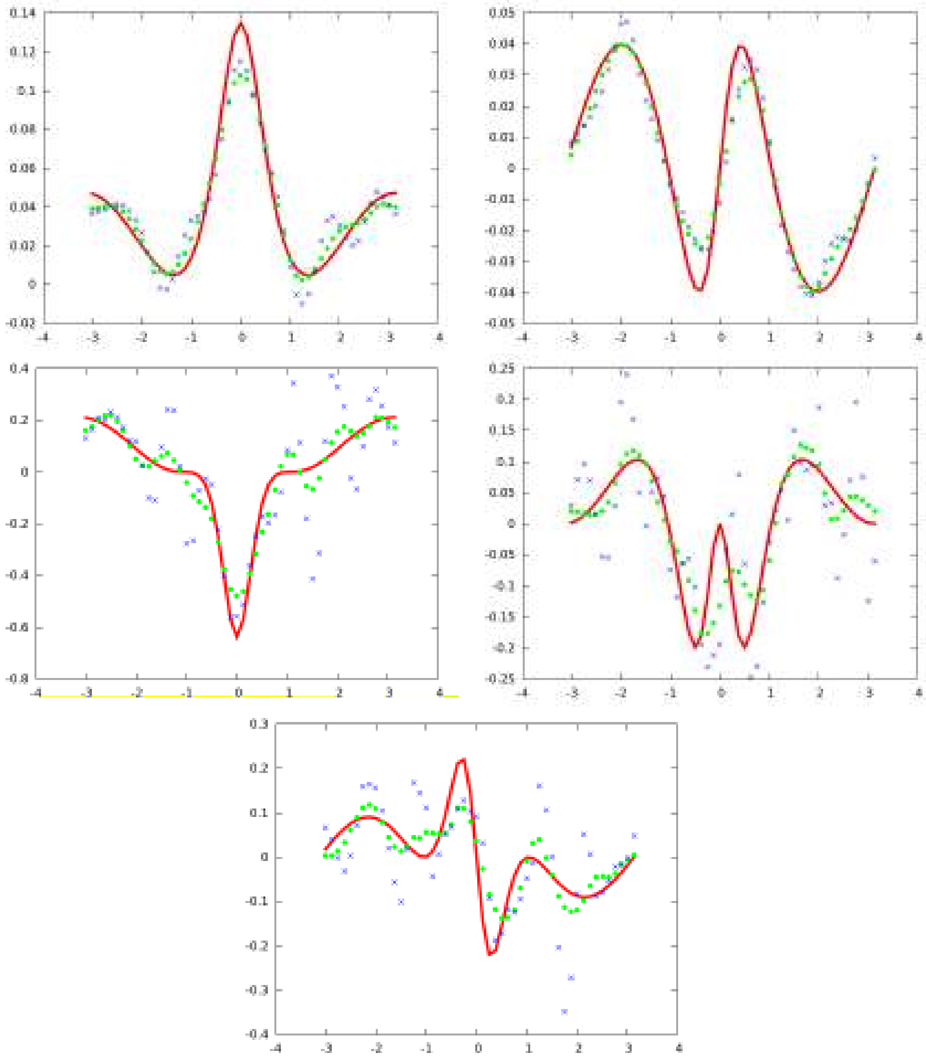

FIGURE 8. Amplitude of noise $\delta_r = 0.1$. Top left: u_1, Top right: u_2, Middle left: σ_{11}, Middle right: σ_{22}, Bottom: σ_{12}

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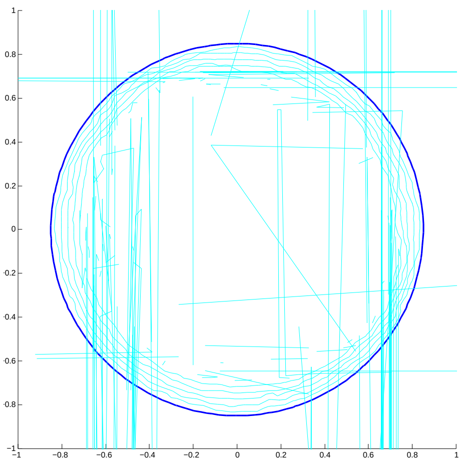
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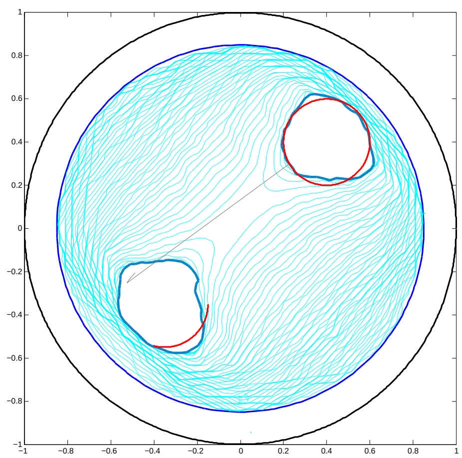

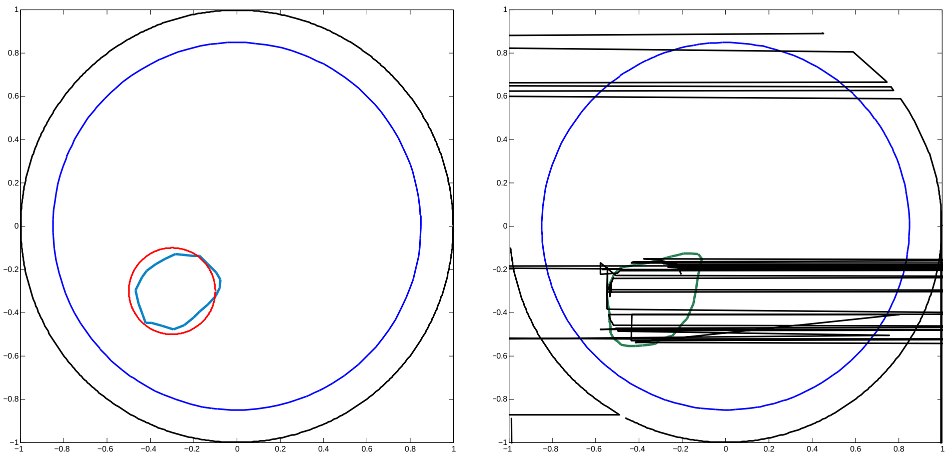
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