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A duality-based method of quasi-reversibility to solve the Cauchy problem in the presence of noisy data

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Abstract

In this paper, we introduce a new version of the method of quasi-reversibility to solve the ill-posed Cauchy problems for the Laplace's equation in the presence of noisy data. It enables one to regularize the noisy Cauchy data and to select a relevant value of the regularization parameter in order to use the standard method of quasi-reversibility. Our method is based on duality in optimization and is inspired by the Morozov's discrepancy principle. Its efficiency is shown with the help of some numerical experiments in two dimensions.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

We consider the Cauchy problem for Laplace's equation in a bounded domain of $\mathbb{R}^N$ ($N \geqslant 2$). It is now well-known that such a problem is ill-posed in the sense of Hadamard. Since the pioneering work of Hadamard himself [17], the ill-posedness for the Cauchy problem was interpreted in several ways, for example, by analyzing the eigenvalues of the Steklov–Poincaré's operator like in [3] or by establishing stability estimates like in [1, 7]. Each of these analysis reveals that the problem is somehow 'exponentially' ill-posed.

Since the Cauchy problem arises in many inverse problems which involve noisy measurements, we need to regularize the Cauchy problem and among all possible regularization techniques (see for example [2, 11] for an overview), the method of quasi-reversibility (QR) introduced in [21] has many interesting features. In particular, due to its variational form, it can be discretized in complex geometries with the help of the finite element method (FEM). Moreover it is non-iterative by nature, that is, the computation of an approximate solution in the FEM context requires only one matrix inversion. The QR method consists in replacing the former second-order ill-posed Cauchy problem into a family of well-posed fourth-order problems that depend on a regularization parameter ε. The solution of quasi-reversibility is close to the exact solution when ε is small, and this is precisely the reason why we can consider the method of quasi-reversibility as a regularization technique. Recently, such method showed very good efficiency to solve the inverse obstacle problem [8], and this kind of success encourages us to improve our understanding of the method.

In our opinion, the method of quasi-reversibility for elliptic ill-posed Cauchy problems raises many questions like:

(i) What type of finite element formulation shall be used?
(ii) In the presence of uncontaminated data, what is the expected convergence rate when ε tends to 0?
(iii) In the presence of noisy data, how shall we treat these non-smooth boundary conditions and choose the regularization parameter ε?

Some partial answers to those questions are already available in the literature. Concerning the first question, a mixed formulation based on classical Lagrange finite elements was proposed in [5], while nonconforming formulations based on Hermite finite elements were proposed in [8, 10]. It should be noted that other kinds of discretizations of quasi-reversibility may be implemented in simple geometries, such as, for example, finite differences [19, 21] or splines [13].

The second question was first addressed in [19], where a Hölder-type convergence rate was proved in a truncated domain. The convergence rate in the whole domain was proved to be logarithmic, for $C^{1,1}$ domains in [7] and for Lipschitz domains in [9].

This paper is devoted to the third question. Since the data are contaminated by some noise, strictly speaking the data are not sufficiently smooth to use the method of quasi-reversibility. Additionally, if δ denotes the amplitude of noise, ε has to be chosen as a function of δ. This fact is highlighted in particular in [8], where however no rigorous method for choosing ε was adopted. It seems to the authors that in spite of the analysis presented in [6] and in [11], this question is still partially open for reasons that are detailed in the next section. This is why our paper is specifically devoted to the treatment of noisy data.

2. Statement of the problem

Let Ω be a bounded, connected open set of $\mathbb{R}^N$ $(N \geqslant 2)$ of class $C^{1,1}$. The Cauchy problem for Laplace's equation consists, from the Cauchy data (g_0, g_1) on Γ, in finding u in Ω such that

$$\begin{cases} \Delta u = 0 \quad \text{in} \quad \Omega \\ u|_\Gamma = g_0 \\ \dfrac{\partial u}{\partial n}|_\Gamma = g_1, \end{cases} \tag{1}$$

where Γ is a non-empty open subset of $\partial\Omega$ and n is the outward normal vector on $\partial\Omega$.

Existence of a solution u solving (1) does not hold in general. However, in the case of existence, u is uniquely defined by (g_0, g_1), as it may be seen for example in [7]. In the following we assume there exists $u \in H^2(\Omega)$ satisfying (1) and u will be referred to as the exact solution. Since Ω is of class $C^{1,1}$ and $u \in H^2(\Omega)$, we have $(g_0, g_1) \in H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$. Defining the sets

$$V_g = \{v \in H^2(\Omega) | v = g_0, \partial_n v = g_1 \text{ on } \Gamma\}$$
$$V_0 = \{v \in H^2(\Omega) | v = 0, \partial_n v = 0 \text{ on } \Gamma\},$$

V_0 is a Hilbert space which can be endowed with the classical norm of $H^2(\Omega)$. We now introduce the same formulation of quasi-reversibility as in [8].

Problem (QR_ε). *Find $u_\varepsilon \in V_g$ such that for all $v \in V_0$, we have*

$$(\Delta u_\varepsilon, \Delta v)_{L^2(\Omega)} + \varepsilon(u_\varepsilon, v)_{H^2(\Omega)} = 0.$$

2.1. The case of smooth data

The QR formulation can be easily generalized to the case of noisy data provided these data are smooth, namely $\left(g_0^\delta, g_1^\delta\right) \in H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$ such that $\left\|g_0^\delta - g_0\right\|_{H^{3/2}(\Gamma)} \leqslant \delta$ and $\left\|g_1^\delta - g_1\right\|_{H^{1/2}(\Gamma)} \leqslant \delta$. To this end we use a continuous extension operator R : $H^{3/2}(\Gamma) \times H^{1/2}(\Gamma) \to H^2(\Omega)$ such that $U = R(g_0, g_1)$ satisfies $U|_\Gamma = g_0$ and $\partial_n U|_\Gamma = g_1$. A proof of existence of such operator R can be found for example in [15].

The following proposition provides the justification of the method.

Proposition 1. *For exact data $(g_0, g_1) \in H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$, the problem (QR_ε) has a unique solution $u_\varepsilon \in V_g$. For noisy data $\left(g_0^\delta, g_1^\delta\right) \in H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$, the problem (QR_ε) has a unique solution $u_\varepsilon^\delta \in V_g^\delta$, where V_g^δ is analogous of V_g with (g_0, g_1) replaced by $\left(g_0^\delta, g_1^\delta\right)$. We have*

$$\lim_{\varepsilon \to 0} \|u_\varepsilon - u\|_{H^2(\Omega)} = 0, \qquad \|u_\varepsilon^\delta - u_\varepsilon\|_{H^2(\Omega)} \leqslant C\frac{\delta}{\sqrt{\varepsilon}},$$

where $C > 0$ is a constant.

The proof of proposition 1 is omitted here since it is very similar to the proofs provided in [5, 6] in slightly different cases. In particular, well-posedness for problem (QR_ε) is based on the change of variable $\hat{u}_\varepsilon = u_\varepsilon - U$ in problem (QR_ε). The following error estimate between the QR solution with noisy data and the exact solution immediately results from proposition 1:

$$\left\|u_\varepsilon^\delta - u\right\|_{H^2(\Omega)} \leqslant r(\varepsilon) + C\frac{\delta}{\sqrt{\varepsilon}}, \qquad \lim_{\varepsilon \to 0} r(\varepsilon) = 0. \tag{2}$$

For obvious reasons, we aim at minimizing the right-hand side. With uncontaminated data ($\delta = 0$), it is clear that the smaller the ε, the better the approximation. But we meet a classical embarrassing configuration with noisy data ($\delta > 0$), as detailed for example in [18]: the first term tends to 0 when ε tends to 0 while the second term explodes when ε tends to 0. In particular, a very small value ε would produce a bad solution and the idea of choosing a value ε that balances the two terms seems natural. But it should be noted that doing so is impossible in practice since the two terms are at best estimated up to a constant that is unknown, as it is also discussed in [11].

Keeping the assumption of smooth data, a natural way to circumvent this issue is to introduce a slight change in formulation (QR_ε) in order to obtain a Tikhonov regularization.

2.2. A Tikhonov regularization

By introducing the change of variable $\hat{u} = u - U$ directly in the Cauchy problem (1) instead of problem (QR_ε) and by setting $f = -\Delta U$, (1) is equivalent to the homogeneous ill-posed problem

$$\begin{cases} \Delta \hat{u} = f \quad \text{in} \quad \Omega \\ \hat{u}|_\Gamma = 0 \\ \dfrac{\partial \hat{u}}{\partial n}|_\Gamma = 0, \end{cases}$$

which can be regularized by the following homogeneous formulation of quasi-reversibility.

Problem (HQR_ε). *Find $\hat{u}_\varepsilon \in V_0$ such that $\forall \hat{v} \in V_0$, we have*

$$(\Delta\hat{u}_\varepsilon, \Delta\hat{v})_{L^2(\Omega)} + \varepsilon(\hat{u}_\varepsilon, \hat{v})_{H^2(\Omega)} = (f, \Delta\hat{v})_{L^2(\Omega)}.$$

Such technique was already used in [6, 11, 21]. The data are now f and if c denotes the norm of the continuous operator $-\Delta\circ R : H^{3/2}(\Gamma)\times H^{1/2}(\Gamma) \to L^2(\Omega)$, with $f^\delta = -(\Delta\circ R)\big(g_0^\delta, g_1^\delta\big)$ we see that the amplitude of the noise affecting f in $L^2(\Omega)$ is simply estimated by $c\delta$.

Remarkably, the formulation of quasi-reversibility (HQR_ε) exactly corresponds to the Tikhonov regularization of the continuous operator $\Delta : V_0 \to L^2(\Omega)$. Therefore, we can adapt the techniques known in the Tikhonov framework such as Morozov's discrepancy principle like in [6] (see also numerical applications in [10]) or the balancing principle like in [11], in order to calibrate ε as a function of the amplitude $c\delta$ of the noise.

However, we can see that this method has a main drawback: the constant of continuity c is unknown in general, which means that it is impossible to properly convert the amplitude of noise δ on Cauchy data (g_0, g_1) to an amplitude of noise on the homogeneous data f. Furthermore, in both formulations (QR_ε) and (HQR_ε), the use of operator R requires the data to belong to rather smooth Sobolev spaces. Since the available data $\big(g_0^\delta, g_1^\delta\big)$ are noisy, it is more reasonable to assume that they belong to $L^2(\Gamma)\times L^2(\Gamma)$ only. These two issues were left aside in [6, 10, 11].

In order to illustrate them more concretely, we borrow from [11] a simple example in the half-plane $\{(x_1, x_2) \in \mathbb{R}^2,\ x_2 > 0\}$. For such geometry, a simple way of choosing R would be to define U as

$$U(x_1, x_2) = g_0(x_1) - x_2 g_1(x_1),$$

so that

$$f(x_1, x_2) = -g_0''(x_1) + x_2 g_1''(x_1).$$

Computing f from L^2 data (g_0, g_1) involves a double numerical differentiation of the data, which is strongly unstable. Even if we regularize this numerical differentiation, there is no way, except by using strong *a priori* regularity assumptions on the data, to exhibit a constant c of continuity for the operator $(g_0, g_1) \to f$.

In the next section we introduce a formulation which enables us to cope with the two issues described above: an unknown constant of continuity for $-(\Delta\circ R)$ and non-smooth noisy data. Precisely, our formulation avoids using an extension operator R and handles directly the original noisy data $\big(g_0^\delta, g_1^\delta\big) \in L^2(\Gamma)\times L^2(\Gamma)$ as they are. Such method is inspired from Morozov's discrepancy principle and uses duality in optimization.

3. A formulation based on duality

3.1. An optimization problem

We introduce the space $Y = L^2(\Gamma)\times L^2(\Gamma)\times L^2(\Omega)$. For $p = (p_0, p_1, p_2) \in Y$ and $q = (q_0, q_1, q_2) \in Y$, we define the scalar product $(p, q)_Y = (p_0, q_0)_{L^2(\Gamma)} + (p_1, q_1)_{L^2(\Gamma)} + (p_2, q_2)_{L^2(\Omega)}$, which makes Y a Hilbert space. We introduce the operator $A : H^2(\Omega) \to Y$ such that

$$Au = (u|_\Gamma, \partial_n u|_\Gamma, \Delta u).$$

The linear operator A satisfies the following properties.

Proposition 2. *The operator A is continuous, injective with dense range.*

Proof. Continuity comes from the classical properties of traces (see [15]). Injectivity amounts to the uniqueness property for the Cauchy problem. It remains to prove that $\overline{\mathrm{Im} A} = Y$. Let $y = (f_0, f_1, f) \in Y$ such that for all $u \in H^2(\Omega)$, $(Au, y)_Y = 0$, that is,

$$\int_\Omega \Delta u f \, \mathrm{d}x + \int_\Gamma u f_0 \, \mathrm{d}\Gamma + \int_\Gamma \frac{\partial u}{\partial n} f_1 \, \mathrm{d}\Gamma = 0. \tag{3}$$

Taking $u = \phi \in C_0^\infty(\Omega)$, we immediately infer that $\Delta f = 0$ in the distributional sense, so that $f \in \{u \in L^2(\Omega),\ \Delta u \in L^2(\Omega)\}$. Since Ω is of class $C^{1,1}$, we can perform the following integration by part (see [15]):

$$\int_\Omega \Delta u f \, \mathrm{d}x = \left\langle \frac{\partial u}{\partial n}, f \right\rangle_{H^{\frac{1}{2}}(\partial\Omega), H^{-\frac{1}{2}}(\partial\Omega)} - \left\langle u, \frac{\partial f}{\partial n} \right\rangle_{H^{\frac{3}{2}}(\partial\Omega), H^{-\frac{3}{2}}(\partial\Omega)} + \int_\Omega u \Delta f \, \mathrm{d}x. \tag{4}$$

We introduce the following notations: $\langle \cdot, \cdot \rangle_{H^s_{00}(\Gamma), H^{-s}(\Gamma)}$ for $s = \frac{1}{2}, \frac{3}{2}$ denotes the duality pairing between $H^s_{00}(\Gamma)$ and $H^{-s}(\Gamma)$. Here $H^s_{00}(\Gamma)$ denotes the space of all $u \in H^s(\Gamma)$ such that the continuation $\tilde{u}$ of u by zero on $\partial\Omega$ outside Γ belongs to $H^s(\partial\Omega)$, and $H^{-s}(\Gamma)$ the set of restrictions on Γ of elements in $H^{-s}(\partial\Omega)$.

Now we take $h_0 \in H^{3/2}_{00}(\Gamma)$ and choose $u \in H^2(\Omega)$ such that $(u|_{\partial\Omega}, \partial_n u|_{\partial\Omega}) = (\tilde{h}_0, 0)$. From (3) and (4) we infer that

$$\int_\Gamma h_0 f_0 \, \mathrm{d}\Gamma = \left\langle h_0, \frac{\partial f}{\partial n} \right\rangle_{H^{\frac{3}{2}}_{00}(\Gamma), H^{-\frac{3}{2}}(\Gamma)}.$$

It follows that $\partial_n f = f_0$ on Γ. Similarly is we now take $h_1 \in H^{1/2}_{00}(\Gamma)$ and choose $u \in H^2(\Omega)$ such that $(u|_{\partial\Omega}, \partial_n u|_{\partial\Omega}) = (0, \tilde{h}_1)$, if follows that $f = -f_1$ on Γ.

By employing the same procedure on the interior of the complementary part of Γ in $\partial\Omega$, which is denoted by Γ_c, we obtain $f = \partial_n f = 0$ on Γ_c. Using uniqueness for $L^2(\Omega)$ functions that satisfy the Cauchy problem in $C^{1,1}$-class domains (see [6], proposition 1), we have $f = 0$ in Ω, and then $f_0 = f_1 = 0$, which completes the proof. □

We consider now some noisy data $\left(g_0^\delta, g_1^\delta\right) \in L^2(\Gamma) \times L^2(\Gamma)$ such that $\left\|g_0^\delta - g_0\right\|_{L^2(\Gamma)} \leqslant \delta$ and $\left\|g_1^\delta - g_1\right\|_{L^2(\Gamma)} \leqslant \delta$ for given δ.

We introduce the following optimization problem.

Problem (P_α).

$$\inf_{v \in C_\alpha} F(v) \qquad \text{where} \quad F(v) = \frac{1}{2}\|v\|^2_{H^2(\Omega)},$$

and the set C_α is defined for $\alpha \geqslant 0$ by

$$C_\alpha = \left\{v \in H^2(\Omega),\ Av \in B\left(g_0^\delta, \delta\right) \times B\left(g_1^\delta, \delta\right) \times B(0, \alpha)\right\}.$$

Here, B(c,r) denotes the closed ball of center c and radius r for the appropriate L^2 norm.

The role of the parameter α will be clarified later.

Well-posedness of problem (P_α) is given by the following proposition.

Proposition 3. *For $\alpha \geqslant 0$, problem (P_α) has a unique solution u_α^δ.*

Proof. The function F is continuous, coercive and strictly convex. The set C_α is non-empty since $u \in C_\alpha$. It is also closed and convex, and the result follows from [14], proposition 1.2, p 34. □

Since our data $\left(g_0^\delta, g_1^\delta\right)$ are noisy, there is no hope to retrieve the unknown exact solution u. For $\alpha = 0$, the solution $\mathrm{u}^\delta := \mathrm{u}_0^\delta$ of problem (P_0) can be viewed as the new reference solution we are looking for in the presence of contaminated data with amplitude δ. Solution u^δ is the harmonic function of the minimal H^2 norm in Ω such that the traces $(\mathrm{u}^\delta|_\Gamma, \partial_n \mathrm{u}^\delta|_\Gamma)$ approach the noisy data (g_0^δ, g_1^δ) up to the noise amplitude δ. In this sense, we follow the well-known Morozov's discrepancy principle.

The following proposition clarifies the behavior of u_α^δ when α tends to 0.

Proposition 4.

$$\lim_{\alpha\to 0} \left\| \mathrm{u}_\alpha^\delta - \mathrm{u}^\delta \right\|_{H^2(\Omega)} = 0.$$

Proof. We have $C_0 \subset C_\alpha$; hence, $\left\|\mathrm{u}_\alpha^\delta\right\|_{H^2(\Omega)} \leqslant \|\mathrm{u}^\delta\|_{H^2(\Omega)}$. It follows that we can extract a subsequence, still denoted by u_α^δ, and find $w \in H^2(\Omega)$ such that $\mathrm{u}_\alpha^\delta \rightharpoonup w$ in $H^2(\Omega)$.

On the one hand, we remark that $\Delta \mathrm{u}_\alpha^\delta \to 0$ in $L^2(\Omega)$, so that $\Delta w = 0$. On the other hand, the sets $B(g_i^\delta, \delta)$ $(i = 0, 1)$ are closed and convex in $L^2(\Gamma)$, so they are weakly closed. Hence, $w|_\Gamma \in B(g_0^\delta, \delta)$ and $\partial_n w|_\Gamma \in B(g_1^\delta, \delta)$. We conclude that $w \in C_0$. By the same argument, we also deduce from $\left\|\mathrm{u}_\alpha^\delta\right\|_{H^2(\Omega)} \leqslant \|\mathrm{u}^\delta\|_{H^2(\Omega)}$ that $\|w\|_{H^2(\Omega)} \leqslant \|\mathrm{u}^\delta\|_{H^2(\Omega)}$. Uniqueness for problem (P_0) implies that $w = \mathrm{u}^\delta$, and then $\mathrm{u}_\alpha^\delta \rightharpoonup \mathrm{u}^\delta$ in $H^2(\Omega)$.

Writing

$$\left\|\mathrm{u}_\alpha^\delta - \mathrm{u}^\delta\right\|_{H^2(\Omega)}^2 \leqslant 2\|\mathrm{u}^\delta\|_{H^2(\Omega)}^2 - 2\left(\mathrm{u}_\alpha^\delta, \mathrm{u}^\delta\right)_{H^2(\Omega)}$$

implies that $\mathrm{u}_\alpha^\delta \to \mathrm{u}^\delta$ in $H^2(\Omega)$. A classical argument by contradiction leads to the global convergence of the sequence u_α^δ. □

Remark 1. We prove approximately the same way that $\lim_{\delta\to 0} \|\mathrm{u}^\delta - u\|_{H^2(\Omega)} = 0$.

3.2. About the dual problem

We begin with redefining problem (P_α) as

$$(P_\alpha) \qquad \inf_{v\in H^2(\Omega)} F(v) + I_\alpha(Av), \tag{5}$$

where I_α is the indicator function of $B_\alpha := B(g_0^\delta, \delta) \times B(g_1^\delta, \delta) \times B(0, \alpha)$, that is,

$$\begin{cases} I_\alpha(y) = 0 & \text{if} \quad y \in B_\alpha \\ I_\alpha(y) = +\infty & \text{if} \quad y \notin B_\alpha. \end{cases}$$

For the sake of self-containment, we recall the theorem of Fenchel–Rockafellar (see theorem 4.1 in [14], p 58).

Theorem (Fenchel–Rockafellar). *Let us denote V and Y as two Hilbert spaces, V^* and Y^* as the corresponding dual spaces. Let us denote $A : V \to Y$ as a continuous operator, and $A^* : Y^* \to V^*$ as its adjoint operator. Finally, let us denote $J : V \times Y \to \overline{\mathbb{R}}$ as a convex, lower semi-continuous and proper function, and $J^* : V^* \times Y^* \to \overline{\mathbb{R}}$ as the Fenchel conjugate function of J.*

We consider the primal minimization problem

$$(P) \qquad \inf_{v\in V} J(v, Av)$$

and the dual maximization problem

$$(P^*) \qquad \sup_{y^*\in Y^*} -J^*(A^*y^*, -y^*).$$

If we have $\inf(P) < +\infty$ *and if there exists* $v_0 \in V$ *such that* $J(v_0, Av_0) < +\infty$ *and* $y \to J(v_0, y)$ *is continuous at* Av_0*, then*

$$\inf(P) = \sup(P^*)$$

and problem P^* *has at least one solution.*

We immediately check that for all $\alpha > 0$ the Fenchel–Rockafellar theorem is applicable with $V = H^2(\Omega)$, Y and A defined at the beginning of the section and J defined for $v \in V$ and $y \in Y$ by

$$J(v, y) = F(v) + I_\alpha(Ay).$$

Furthermore, we identify the space Y^* to Y itself.

By following the same calculations as in [6], the dual problem (P_α^*) can be written as

$$(P_\alpha^*) \quad \sup_{q\in Y} -(F^*(A^*q) + I_\alpha^*(-q)) \tag{6}$$

and then we arrive at

Problem (P_α^*)**.**

$$\inf_{q\in Y} G_\alpha(q)$$

with

$$G_\alpha(q) = \tfrac{1}{2}\|A^*q\|^2_{H^2(\Omega)} + \delta\|q_0\|_{L^2(\Gamma)} + \delta\|q_1\|_{L^2(\Gamma)} + \alpha\|q_2\|_{L^2(\Omega)} - \left(g_0^\delta, q_0\right)_{L^2(\Gamma)} - \left(g_1^\delta, q_1\right)_{L^2(\Gamma)}.$$

Concerning the dual problem (P_α^*), we have the following proposition.

Proposition 5. *For* $\alpha > 0$*, problem* (P_α^*) *has a unique solution* $p_\alpha^\delta = \left(p_{\alpha,0}^\delta, p_{\alpha,1}^\delta, p_{\alpha,2}^\delta\right)$.

Proof. The claim results from the following properties of function G_α: G_α is continuous, strictly convex and coercive. The first property is obvious. Concerning the second one, convexity is obvious while the strict convexity follows from the injectivity of A^* due to proposition 2 (in particular A has a dense range). It remains to prove that G_α is coercive. Assume it is not, that is, there exists a sequence of $q_n \in Y$ such that $\|q_n\|_Y \to +\infty$ and $G_\alpha(q_n) \leqslant C$ for some constant $C > 0$. Let us define $\beta_n = \|q_n\|_Y$ and $p_n = q_n/\beta_n = (p_{n,0}, p_{n,1}, p_{n,2})$.

We have

$$\frac{G_\alpha(q_n)}{\beta_n^2} = \frac{1}{2}\|A^*p_n\|^2_{H^2(\Omega)} + \frac{\delta}{\beta_n}(\|p_{n,0}\|_{L^2(\Gamma)} + \|p_{n,1}\|_{L^2(\Gamma)}) + \frac{\alpha}{\beta_n}\|p_{n,2}\|_{L^2(\Omega)} - \frac{1}{\beta_n}\left(\left(g_0^\delta, p_{n,0}\right)_{L^2(\Gamma)} + \left(g_1^\delta, p_{n,1}\right)_{L^2(\Gamma)}\right),$$

and then

$$\frac{1}{2}\|A^*p_n\|^2_{H^2(\Omega)} \leqslant \frac{C}{\beta_n^2} + \frac{1}{\beta_n}\left(\left\|g_0^\delta\right\|_{L^2(\Gamma)} + \left\|g_1^\delta\right\|_{L^2(\Gamma)}\right).$$

We conclude that $A^*p_n \to 0$ when $n \to +\infty$.

On the other hand, since $\|p_n\|_Y = 1$, there exists a subsequence of (p_n) still denoted by (p_n) and $p \in Y$ such that $p_n \rightharpoonup p$ in Y. It follows that $A^*p = 0$, so that $p = 0$.

We have

$$G_\alpha(q_n) \geqslant \beta_n\big(\delta\|p_{n,0}\|_{L^2(\Gamma)} + \delta\|p_{n,1}\|_{L^2(\Gamma)} + \alpha\|p_{n,2}\|_{L^2(\Omega)} - \left(g_0^\delta, p_{n,0}\right)_{L^2(\Gamma)} - \left(g_1^\delta, p_{n,1}\right)_{L^2(\Gamma)}\big),$$

that is,

$$G_\alpha(q_n) \geqslant \beta_n\big(\min(\alpha,\delta) - \big(g_0^\delta, p_{n,0}\big)_{L^2(\Gamma)} - \big(g_1^\delta, p_{n,1}\big)_{L^2(\Gamma)}\big),$$

so that $G_\alpha(q_n) \to +\infty$ when $n \to +\infty$, which is a contradiction. □

The solutions of the primal and dual problems (P_α) and (P_α^*) are related in the following proposition.

Proposition 6. *For $\alpha > 0$, let u_α^δ and p_α^δ denote the solutions of problems (P_α) and (P_α^*), respectively. Then $\mathrm{u}_\alpha^\delta = A^* p_\alpha^\delta$.*

Proof. For the sake of simplicity, we omit the parameter δ in notations u_α^δ and p_α^δ. By applying the Fenchel–Rockafellar theorem, we obtain that

$$\inf(P_\alpha) = \sup(P_\alpha^*) < +\infty.$$

It follows in view of (5) and (6) that

$$F(\mathrm{u}_\alpha) + I_\alpha(A\mathrm{u}_\alpha) = -F^*(A^*p_\alpha) - I_\alpha^*(-p_\alpha) = -G_\alpha(p_\alpha) < +\infty;$$

hence,

$$\{F(\mathrm{u}_\alpha) + F^*(A^*p_\alpha) - (\mathrm{u}_\alpha, A^*p_\alpha)_{H^2(\Omega)}\} + \{I_\alpha(A\mathrm{u}_\alpha) + I_\alpha^*(-p_\alpha) - (A\mathrm{u}_\alpha, -p_\alpha)_Y\} = 0.$$

From the definition of the conjugate function, both terms of the sum are positive and then vanish. In particular

$$F(\mathrm{u}_\alpha) + F^*(A^*p_\alpha) - (\mathrm{u}_\alpha, A^*p_\alpha)_{H^2(\Omega)} = 0,$$

and from proposition 5.1 in [14], p 21, it follows that A^*p_α belong to the subdifferential of F at u_α, that is, $A^*p_\alpha = \mathrm{u}_\alpha$. □

Remark 2. Duality in optimization enables us to transform the constrained minimization problem (P_α) into the unconstrained minimization (P_α^*). Problem (P_α^*) is hence easier to solve than problem (P_α).

Remark 3. It is natural to wonder what happens when $\alpha = 0$. The Fenchel–Rockafellar is not applicable any more because the set $C_0 = \big\{v \in H^2(\Omega),\ Av \in B\big(g_0^\delta, \delta\big) \times B\big(g_1^\delta, \delta\big) \times \{0\}\big\}$ has an empty interior.

Nevertheless, it is possible to invert the role of primal problem (P_0) and dual problem (P_0^*). Actually, for all $q \in Y$, in view of the proof of proposition 6 we have for $\alpha > 0$

$$-G_\alpha(q) \leqslant -G_\alpha(p_\alpha) = \tfrac{1}{2}\big\|\mathrm{u}_\alpha^\delta\big\|_{H^2(\Omega)}^2.$$

By passing to the limit $\alpha \to 0$ and by using proposition 4, it follows that for all $q \in Y$,

$$-G_0(q) \leqslant \tfrac{1}{2}\|\mathrm{u}^\delta\|_{H^2(\Omega)}^2$$

and hence $\sup(P_0^*) < +\infty$. Since $-G_0$ is a continuous function, we can apply the Fenchel–Rockafellar theorem by considering (P_0^*) as the primal problem and (P_0) as its dual problem and then for $\alpha = 0$ we have again

$$\inf(P_0) = \sup(P_0^*) < +\infty,$$

that is,

$$\inf_{q\in Y} G_0(q) = -\tfrac{1}{2}\|\mathrm{u}^\delta\|_{H^2(\Omega)}^2.$$

But the problem is that G_0 is not coercive any more (see the proof of proposition 5), so that we cannot prove the well-posedness of problem (P_0^*). In other words, proposition 5 does not hold any more, nor proposition 6. This is the reason why we have introduced $\alpha > 0$ and the relaxed problems (P_α).

3.3. Going back to quasi-reversibility

We now establish a relationship between the solution u_α^δ of the optimization problem (P_α) and the solution of a QR problem. We have the following proposition.

Proposition 7. *In the case $\|g_0^\delta\|_{L^2(\Gamma)} > \delta$ or $\|g_1^\delta\|_{L^2(\Gamma)} > \delta$, for sufficiently small $\alpha > 0$, we have $p_{\alpha,2}^\delta \neq 0$ and u_α^δ is the solution of problem (QR_ε) with Cauchy data $\left(\mathrm{u}_\alpha^\delta|_\Gamma, \partial_n \mathrm{u}_\alpha^\delta|_\Gamma\right)$ and*

$$\varepsilon = \frac{\alpha}{\|p_{\alpha,2}^\delta\|_{L^2(\Omega)}}. \tag{7}$$

In the case $\|g_0^\delta\|_{L^2(\Gamma)} \leqslant \delta$ and $\|g_1^\delta\|_{L^2(\Gamma)} \leqslant \delta$, we have $\mathrm{u}^\delta = \mathrm{u}_\alpha^\delta = 0$ and $p_\alpha^\delta = 0$.

Proof. Again we omit the parameter δ in notations u_α^δ and p_α^δ.

In the second case $\|g_0^\delta\|_{L^2(\Gamma)} \leqslant \delta$ and $\|g_1^\delta\|_{L^2(\Gamma)} \leqslant \delta$, it is straightforward that $\mathrm{u}^\delta = 0$ and $\mathrm{u}_\alpha = 0$ and then in view of proposition 6 and injectivity of A^* that $p_\alpha = 0$.

Now we consider the case $\|g_0^\delta\|_{L^2(\Gamma)} > \delta$ or $\|g_1^\delta\|_{L^2(\Gamma)} > \delta$. First we prove that for sufficiently small α we have $p_{\alpha,2} \neq 0$.

Assuming this is not true, we could find some sequence (α_n) such that $\alpha_n \to 0$ and $p_{\alpha_n,2} = 0$. For $\phi \in C_0^\infty(\Omega)$, we have

$$(p_{\alpha_n}, A\phi)_Y = (p_{\alpha_n,2}, \Delta\phi)_{L^2(\Omega)} + (p_{\alpha_n,0}, \phi)_{L^2(\Gamma)} + (p_{\alpha_n,1}, \partial_n\phi)_{L^2(\Gamma)} = 0.$$

On the other hand we have

$$(p_{\alpha_n}, A\phi)_Y = (A^* p_{\alpha_n}, \phi)_{H^2(\Omega)} = (\mathrm{u}_{\alpha_n}, \phi)_{H^2(\Omega)}.$$

By passing to the limit $n \to +\infty$, we arrive at $(\mathrm{u}^\delta, \phi)_{H^2(\Omega)} = 0$ for all $\phi \in C_0^\infty(\Omega)$, and then $\Delta\Delta\mathrm{u}^\delta - \Delta\mathrm{u}^\delta + \mathrm{u}^\delta = 0$ in the distributional sense, that is, $\mathrm{u}^\delta = 0$ since $\Delta\mathrm{u}^\delta = 0$ in Ω. We conclude that $\|g_0^\delta\|_{L^2(\Gamma)} \leqslant \delta$ and $\|g_1^\delta\|_{L^2(\Gamma)} \leqslant \delta$, which is a contradiction.

Second, we prove that for sufficiently small α we have

$$\Delta\mathrm{u}_\alpha = -\alpha \frac{p_{\alpha,2}}{\|p_{\alpha,2}\|_{L^2(\Omega)}}.$$

By differentiation of G_α at p_α following the direction $q = (0, 0, q_2) \in Y$, we obtain that for all $q_2 \in L^2(\Omega)$,

$$(A^* p_\alpha, A^* q)_{H^2(\Omega)} + \frac{\alpha}{\|p_{\alpha,2}\|_{L^2(\Omega)}}(p_{\alpha,2}, q_2)_{L^2(\Omega)} = 0.$$

On the other hand,

$$(A^* p_\alpha, A^* q)_{H^2(\Omega)} = (\mathrm{u}_\alpha, A^* q)_{H^2(\Omega)} = (A\mathrm{u}_\alpha, q)_Y = (\Delta\mathrm{u}_\alpha, q_2)_{L^2(\Omega)},$$

and the result follows.

Lastly, for sufficiently small α and for all $v \in V_0$, we have

$$\begin{aligned}(\Delta\mathrm{u}_\alpha, \Delta v)_{L^2(\Omega)} &= -\left(\alpha \frac{p_{\alpha,2}}{\|p_{\alpha,2}\|_{L^2(\Omega)}}, \Delta v\right)_{L^2(\Omega)} \\ &= -\frac{\alpha}{\|p_{\alpha,2}\|_{L^2(\Omega)}}(p_\alpha, Av)_Y = 0 \\ &= -\frac{\alpha}{\|p_{\alpha,2}\|_{L^2(\Omega)}}(\mathrm{u}_\alpha, v)_{H^2(\Omega)},\end{aligned}$$

which is the claimed result by uniqueness of the QR solution. □

Remark 4. From the proof of proposition 6, it follows that if $\|g_0^\delta\|_{L^2(\Gamma)} > \delta$ or $\|g_1^\delta\|_{L^2(\Gamma)} > \delta$, then for sufficiently small $\alpha > 0$

$$\Delta \mathrm{u}_\alpha^\delta = -\alpha \frac{p_{\alpha,2}^\delta}{\|p_{\alpha,2}\|_{L^2(\Omega)}};$$

in particular,

$$\|\Delta \mathrm{u}_\alpha^\delta\|_{L^2(\Omega)} = \alpha.$$

If we assume that $p_{\alpha,0}^\delta \neq 0$ and $p_{\alpha,1}^\delta \neq 0$, by differentiation of G_α at p_α^δ following the directions $q = (q_0, 0, 0) \in Y$ and $q = (0, q_1, 0) \in Y$, we immediately obtain that

$$\mathrm{u}_\alpha^\delta|_\Gamma = g_0^\delta - \delta \frac{p_{\alpha,0}^\delta}{\|p_{\alpha,0}^\delta\|_{L^2(\Gamma)}}, \qquad \partial_n \mathrm{u}_\alpha^\delta|_\Gamma = g_1^\delta - \delta \frac{p_{\alpha,1}^\delta}{\|p_{\alpha,1}^\delta\|_{L^2(\Gamma)}};$$

in particular,

$$\|\mathrm{u}_\alpha^\delta|_\Gamma - g_0^\delta\|_{L^2(\Gamma)} = \delta, \qquad \|\partial_n \mathrm{u}_\alpha^\delta|_\Gamma - g_1^\delta\|_{L^2(\Gamma)} = \delta.$$

We can interpret the Cauchy data $(\mathrm{u}_\alpha^\delta|_\Gamma, \partial_n \mathrm{u}_\alpha^\delta|_\Gamma)$ in $H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$ as regularized data obtained from the noisy Cauchy data (g_0^δ, g_1^δ) in $L^2(\Gamma) \times L^2(\Gamma)$ and corrected with the help of solution p_α^δ.

Remark 5. It should be remarked that our approach is still valuable if we take into account two different amplitudes of noise δ_0 and δ_1 affecting data g_0 and g_1, respectively, that is, $\delta = (\delta_0, \delta_1)$ with

$$\|g_0^\delta - g_0\|_{L^2(\Gamma)} \leqslant \delta_0, \qquad \|g_1^\delta - g_1\|_{L^2(\Gamma)} \leqslant \delta_1.$$

Remark 6. In the section devoted to numerical results hereafter, we use the formulation based on duality in a polygonal domain of $\mathbb{R}^2$, which is not a $C^{1,1}$ domain. However, it can be proved that the results of section 3 still hold in this context, namely propositions 2–7. The main change concerns the proof of proposition 2, where the Green's formula (4) has to be replaced by the one given in [16], theorem 1.5.3, p 26.

3.4. A strategy to solve the Cauchy problem in the presence of noisy data

We are in a position to propose a new strategy to solve the Cauchy problem in the presence of noisy data $(g_0^\delta, g_1^\delta) \in L^2(\Gamma) \times L^2(\Gamma)$, with the assumptions $\|g_i^\delta - g_i\|_{L^2(\Gamma)} \leqslant \delta$, $i = 0, 1$, where δ is the known amplitude of noise.

With exact data, that is, $\delta = 0$, we have $u_\varepsilon \to u$ when $\varepsilon \to 0$ and therefore the solution u_ε of problem (QR_ε) has to be computed with the 'smallest' possible ε. In the continuous setting, such value is meaningless but in the discretized setting, we choose the smallest ε that is compatible with numerical inversion.

With contaminated data, that is $\delta > 0$, there is no hope to retrieve the exact solution u by solving problem (QR_ε) from the noisy data and we do not know how to choose ε. In particular, the 'smallest' possible ε is a bad choice, as discussed previously.

A reasonable objective is to find the solution $\mathrm{u}^\delta = \mathrm{u}_0^\delta$ of problem (P_0) instead of u, in accordance to Morozov's discrepancy principle: boundary conditions are not satisfied exactly but up to δ, which is the known amplitude of noise. More explicitly, u^δ is the solution of

$$\inf_{v \in H^2(\Omega)} \|v\|_{H^2(\Omega)}$$

with constraints

$$\Delta v = 0 \quad \text{in} \quad \Omega, \qquad \left\| v|_\Gamma - g_0^\delta \right\|_{L^2(\Gamma)} \leqslant \delta, \qquad \left\| \partial_n v|_\Gamma - g_1^\delta \right\|_{L^2(\Gamma)} \leqslant \delta.$$

In order to use duality in optimization, we introduce a small parameter $\alpha > 0$ and rather than u_0^δ we consider the solution u_α^δ of the primal problem (P_α). Indeed, u_α^δ is given by $\mathrm{u}_\alpha^\delta = A^* p_\alpha^\delta$, where p_α^δ is the solution of the unconstrained dual problem (P_α^*). Since $\mathrm{u}_\alpha^\delta \to \mathrm{u}^\delta$ when $\alpha \to 0$, u_α^δ is computed with the 'smallest' possible α, that is, in practice the smallest α that is compatible with the numerical inversion.

In short, u_α^δ plays the same role with respect to u^δ in the presence of contaminated data as u_ε with respect to u in the presence of uncontaminated data.

The main interesting feature of our novel version of quasi-reversibility is provided by proposition 7 and remark 4: it enables us to circumvent two issues concerning the standard formulation (QR_ε) of quasi-reversibility. On the one hand, the traces $\left(\mathrm{u}_\alpha^\delta|_\Gamma, \partial_n \mathrm{u}_\alpha^\delta|_\Gamma\right) \in H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$ are regularized Cauchy data obtained from the noisy data $\left(g_0^\delta, g_1^\delta\right) \in L^2(\Gamma) \times L^2(\Gamma)$. On the other hand, u_α^δ coincides with the solution of problem (QR_ε) by selecting $\varepsilon = \varepsilon(\alpha, \delta)$ given by (7) and by prescribing regularized Cauchy data $\left(\mathrm{u}_\alpha^\delta|_\Gamma, \partial_n \mathrm{u}_\alpha^\delta|_\Gamma\right)$ instead of $\left(g_0^\delta, g_1^\delta\right)$, which means that the regularization parameter ε in problem (QR_ε) is determined with the help of our duality-based approach.

The major drawback of our strategy is that computation of u_α^δ is heavier than the computation of u_ε. Problem (P_α^*) is actually a non-quadratic minimization problem, which has to be solved by an iterative algorithm, while problem (QR_ε) needs only one computation. However, it should be noted that in the framework of a free boundary problem like in [8], where we have to solve many Cauchy problems in increasing domains Ω_m ($m \in \mathbb{N}$), it may be time saving to compute first u_α^δ in the initial domain Ω_0, and then to compute the Cauchy data $\left(\mathrm{u}_\alpha^\delta|_\Gamma, \partial_n \mathrm{u}_\alpha^\delta|_\Gamma\right)$ and $\varepsilon(\alpha, \delta)$, which will be our new inputs for the standard problems (QR_ε) in the further steps with updated domains Ω_m for $m \geqslant 1$. In short, a heavy problem is solved only at the first iteration. The efficiency of such procedure is shown in subsection 4.3 of this paper.

4. Numerical results

4.1. Discretization

In order to solve problem (QR_ε) in any polygonal domain Ω of $\mathbb{R}^2$, we use a finite-element method. Since problem (QR_ε) is the weak formulation associated with a fourth-order problem, we need more complex finite elements than classical Lagrange finite elements. Here, we use the so-called Fraeijs de Veubeke's finite element (FV1) introduced in [22], which belongs to the class of nonconforming finite elements for plate problems. Such finite elements are discussed from a mathematical point of view in [12, 20]. In this paper, we briefly describe the FV1 element, but we refer to [8] for a comprehensive analysis of the discretized formulation including convergence analysis. The same finite element will enable us to approximate problems (P_α) and (P_α^*).

Discretization is based on a regular triangulation $\mathcal{T}_h$ of $\overline{\Omega}$ (see [12] for definition) such that the diameter of each triangle $K \in \mathcal{T}_h$ is bounded by h, which is hence the size of the mesh. The set $\overline{\Gamma}$ consists of the union of the edges of some triangles $K \in \mathcal{T}_h$.

The analogous of space $H^2(\Omega)$ in the discretized setting is denoted by V_h, defined as follows. The space V_h is the set of functions $v_h \in L^2(\Omega)$ such that for all $K \in \mathcal{T}_h$, $v_h|_K$ belongs to the space P_K of shape functions in K, and such that the degrees of freedom coincide, namely: the values of the function at the vertices, the values at the mid-points of the edges and the mean values of the normal derivative along each edge. The space P_K is strictly included in the space of degree 3 polynomials and we refer to [20] for exact definition of P_K. The space V_h is equipped with the norm $\|\cdot\|_h$ defined by

$$\|v_h\|_h^2 = \sum_{K\in\mathcal{T}_h} \|v_h\|_{H^2(K)}^2.$$

The finite-dimensional space V_h is not included in $H^2(\Omega)$ (and not even in $H^1(\Omega)$), that is why FV1 belongs to the class of nonconforming finite elements. Then, we define $V_{h,0}$ as the subset of functions of V_h for which the degrees of freedom on the edges contained in $\overline{\Gamma}$ vanish, and $V_{h,g}$ as the subset of functions of V_h for which the degrees of freedom on the edges contained in $\overline{\Gamma}$ coincide with the corresponding values obtained with data g_0 and g_1.

With the help of the above definitions of $V_{h,0}$ and $V_{h,g}$ we are in a position to introduce for $\varepsilon > 0$ the following discretized formulation of quasi-reversibility, written in the weak form.

Problem ($QR_{\varepsilon,h}$). *Find $u_{h,\varepsilon} \in V_{h,g}$, such that for all $v_h \in V_{h,0}$,*

$$\sum_{K\in\mathcal{T}_h} \{(\Delta u_{h,\varepsilon}, \Delta v_h)_{L^2(K)} + \varepsilon(u_{h,\varepsilon}, v_h)_{H^2(K)}\} = 0. \tag{8}$$

The analysis of well-posedness of problem $(QR_{\varepsilon,h})$ and of convergence of the discretized solution $u_{h,\varepsilon}$ of problem $(QR_{\varepsilon,h})$ to the continuous solution u_ε of problem (QR_ε) when the mesh size h tends to 0 is provided in [8].

Now we introduce discretized versions of problems (P_α) and (P_α^*). First we have to define a discretized version of operator $A : H^2(\Omega) \to L^2(\Omega)\times L^2(\Gamma)\times L^2(\Gamma)$ defined for $v \in H^2(\Omega)$ by $Av = (v|_\Gamma, \partial_n v|_\Gamma, \Delta v)$. In this view, we introduce $A_h : V_h \to Y_h := P_{0h} \times P_{1h} \times P_{2h}$ such that for $v_h \in V_h$, $A_h v_h = (\gamma_{0h}(v_h), \gamma_{1h}(v_h), \Delta_h v_h)$. Here spaces P_{0h}, P_{1h}, P_{2h} are defined by the following.

- P_{0h} denotes the set of functions $p_{0h} \in L^2(\Gamma)$ such that p_{0h} is continuous and for all edge $e \in \overline{\Gamma}$, $p_{0h}|_e$ is a degree 2 polynomial;
- P_{1h} denotes the set of functions $p_{1h} \in L^2(\Gamma)$ such that for all edge $e \in \overline{\Gamma}$, $p_{1h}|_e$ is a degree 0 polynomial;
- P_{2h} denotes the set of functions $p_{2h} \in L^2(\Omega)$ such that for all $K \in \mathcal{T}_h$, $p_{2h}|_K$ is a degree 1 polynomial

and operators γ_{0h}, γ_{1h}, Δ_h are defined by the following:

- $\gamma_{0h}(v_h)$ is the function of P_{0h} such that for all $e \in \overline{\Gamma}$, $\gamma_{0h}(v_h)|_e$ is the degree 2 polynomial which coincides with $v_h|_e$ at the end points and at the mid-point of edge e;
- $\gamma_{1h}(v_h)$ is the function of P_{1h} such that for all $e \in \overline{\Gamma}$, $\gamma_{1h}(v_h)|_e$ is the degree 0 polynomial which coincides with the mean value of the exterior normal derivative of v_h along e;
- $\Delta_h v_h$ is the function of P_{2h} that coincides, for all $K \in \mathcal{T}_h$, with Δv_h on K.

Remark 7. The reader may wonder why we did not define γ_{0h} and γ_{1h} simply as the trace of v_h and as the trace of the normal derivative of v_h respectively. Our choice is simpler because it is exactly adapted to the degrees of freedom of our FV1 finite element, namely: any edge of $\overline{\Gamma}$ involves three degrees of freedom of Lagrange type (resp. one degree of freedom of Hermite type), which uniquely determines a degree 2 polynomial (resp. degree 0 polynomial).

With the help of the above definitions, we introduce the discretized primal and dual problems, based on approximations $\left(g_{0,h}^\delta, g_{1,h}^\delta\right) \in P_{0h} \times P_{1h}$ of our noisy data $\left(g_0^\delta, g_1^\delta\right) \in L^2(\Gamma) \times L^2(\Gamma)$.

Problem ($P_{\alpha,h}$).

$$\inf_{v_h \in C_{\alpha,h}} \tfrac{1}{2}\|v_h\|_h^2$$

with

$$C_{\alpha,h} = \left\{v_h \in V_h,\, A_h v_h \in B\left(g_{0,h}^\delta, \delta\right) \times B\left(g_{1,h}^\delta, \delta\right) \times B(0,\alpha)\right\}.$$

Problem ($P_{\alpha,h}^*$).

$$\inf_{q_h \in Y_h} G_{\alpha,h}(q_h)$$

with

$$G_{\alpha,h}(q_h) = \tfrac{1}{2}\|A_h^* q_h\|_h^2 + \delta\|q_{0,h}\|_{L^2(\Gamma)} + \delta\|q_{1,h}\|_{L^2(\Gamma)} + \alpha\|q_{2h}\|_{L^2(\Omega)} - \left(g_{0,h}^\delta, q_{0,h}\right)_{L^2(\Gamma)} - \left(g_{1,h}^\delta, q_{1,h}\right)_{L^2(\Gamma)}.$$

The solutions to problems $(P_{\alpha,h})$ and $(P_{\alpha,h}^*)$ are denoted as $\mathrm{u}_{\alpha,h}^\delta$ and $p_{\alpha,h}^\delta$, respectively, and they are related to each other by the relationship

$$\mathrm{u}_{\alpha,h}^\delta = A_h^* p_{\alpha,h}^\delta. \tag{9}$$

It is easy to prove that well-posedness of problems (P_α) and (P_α^*) and relationship (9) hold provided the operator A_h is onto. It happens that such assumption is true for all our numerical experiments.

In practice, we solve the minimization problem $(P_{\alpha,h}^*)$ by using a gradient method to obtain $p_{\alpha,h}^\delta$, more precisely a limited memory BFGS algorithm (see [4], section 5.3), and then we obtain $\mathrm{u}_{\alpha,h}^\delta$ by using (9).

4.2. Numerical experiments

In the following experiments, the domain Ω is the square $]-0.5, 0.5[\times]-0.5, 0.5[$ in $\mathbb{R}^2$, and the Cauchy data are given on $\Gamma =]-0.5, 0.5[\times\{-0.5\}\cup]-0.5, 0.5[\times\{0.5\}$. The artificial Cauchy data (g_0, g_1) that we consider on Γ are computed from the harmonic function $u(x, y) = -yx^2 + \frac{1}{3}y^3$, which will be referred to as the exact solution.

The domain Ω is triangulated by first dividing each side of the square into segments of equal length h, with $h = 1/70$. However, the triangulation of Ω is unstructured. In order to introduce some noisy data we consider now $g_{0,h}$ and $g_{1,h}$ as vectors, the components of which coincide with the degrees of freedom of P_{0h} and P_{1h}. These components are subjected pointwise to some Gaussian noise, namely

$$g_{0,h}^\delta = g_{0,h} + \sigma\frac{\|g_{0,h}\|_{L^2}}{\|b_{0,h}\|_{L^2}} b_{0,h}, \qquad g_{1,h}^\delta = g_{1,h} + \sigma\frac{\|g_{1,h}\|_{L^2}}{\|b_{1,h}\|_{L^2}} b_{1,h}, \tag{10}$$

where $b_{0,1}$ and $b_{1,h}$ are given by a standard normal distribution, $\sigma > 0$ is a scaling factor and $\|\cdot\|_{L^2}$ denotes the L^2 norm in space P_{0h} (resp. P_{1h}) for $g_{0,h}$ and $b_{0,h}$ (resp. for $g_{1,h}$ and $b_{1,h}$). Obviously, such definition implies that the Cauchy data (g_0, g_1) are contaminated by some relative error of amplitude σ in the L^2 norm. This implies that we have used two different absolute errors of amplitude δ_0 and δ_1 contaminating g_0 and g_1 respectively (see remark 5), with

$$\delta_0 = \sigma\|g_0\|_{L^2}, \qquad \delta_1 = \sigma\|g_1\|_{L^2}.$$

First of all, we illustrate in figure 1 the error estimate (2) for the method of quasi-reversibility with noisy data. To do that we plot $\left\|u_{\varepsilon,h}^\delta - \pi_h u\right\|_h$ as a function of ε, where $u_{\varepsilon,h}^\delta$ is the solution

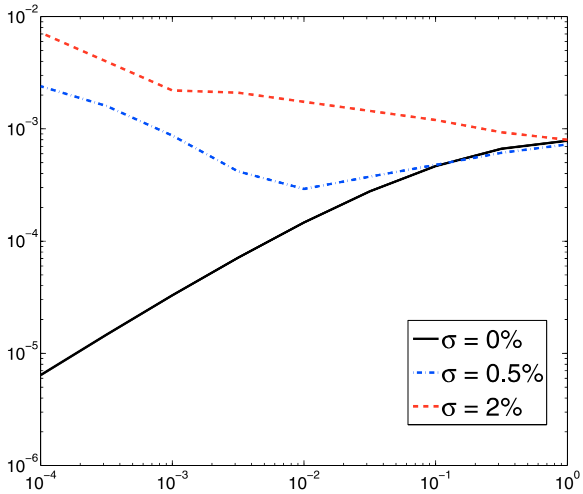

Figure 1. For various σ, $\|u^{\delta}_{\varepsilon,h} - \pi_h u\|_h$ as a function of ε.

of problem $(QR_{\varepsilon,h})$ and $\pi_h u$ denotes the interpolate of the exact solution u in V_h. We observe that with exact data ($\sigma = 0$), the error decreases to 0 when ε decreases to 0, while with noisy data of relative amplitude $\sigma = 0.5\%$, the error first decreases when ε ranges from 1 to 10^{-2}, but then increases when ε goes below 10^{-2}, which is consistent with (2). With noisy data of relative amplitude $\sigma = 2\%$, the error keeps increasing when ε decreases, which means that for large δ, the first term on the right-hand side of (2) is absorbed by the second one. These observations emphasize the need for an alternative approach of problem (QR_ε).

To illustrate the interest of the duality-based approach presented in our paper, we compare the solution $\mathrm{u}^{\delta}_{\alpha,h}$ of problem $(P_{\alpha,h})$ and the solution of the same problem with very small α, which is denoted by u^{δ}_{h}. In practice we can take $\alpha = 0$ to solve problem $(P^{*}_{\alpha,h})$ and stop the iterations when the gradient almost vanishes. As can be seen in the left part of figure 2 with $\sigma = 2\%$, the error between $\mathrm{u}^{\delta}_{\alpha,h}$ and u^{δ}_{h} is decreasing when α is decreasing, which is consistent with proposition 4. As can be seen in the right part of figure 2, the error between $\mathrm{u}^{\delta}_{\alpha,h}$ and $\pi_h u$ is also decreasing when α is decreasing, but there is some incompressible error due to the discrepancy between u^{δ}_{h} and $\pi_h u$, which is independent of α. As explained in subsection 3.4, this shows that we might take α as small as possible. We choose $\alpha = 10^{-4}$ in the following, because it seems to us that for many exact solutions u, the error $\left\|\mathrm{u}^{\delta}_{\alpha,h} - \pi_h u\right\|_h$ is almost stationary when α decreases below 10^{-4}, like in the right part of figure 2. This is, for example, confirmed by the same curves obtained with another function $u(x, y) = \frac{1}{50}\cos(3\pi x)\sinh(3\pi y)$ and represented in figure 3.

Now we want to emphasize the relationship between problem (QR_ε) and problem (P_α) described by proposition 7 and remark 4. We have seen that solving (P^{*}_{α}) and then (P_α) enables us to regularize the noisy Cauchy data $\left(g^{\delta}_{0}, g^{\delta}_{1}\right)$ and to compute a relevant $\varepsilon(\alpha, \delta)$ in the method of quasi-reversibility (QR_ε). In order to illustrate the first point we have plotted on figure 4 (resp. figure 5) the exact Cauchy data (g_0, g_1), the noisy Cauchy data $\left(g^{\delta}_{0,h}, g^{\delta}_{1,h}\right)$ for $\sigma = 2\%$ (resp. $\sigma = 10\%$), as well as the traces $\left(\gamma_{0h}\left(\mathrm{u}^{\delta}_{\alpha,h}\right), \gamma_{1h}\left(\mathrm{u}^{\delta}_{\alpha,h}\right)\right)$ evaluated from the solution $\mathrm{u}^{\delta}_{\alpha,h}$ of problem $(P_{\alpha,h})$. As explained in remark 4, the pair $\left(\mathrm{u}^{\delta}_{\alpha}\big|_{\Gamma}, \partial_n \mathrm{u}^{\delta}_{\alpha}\big|_{\Gamma}\right)$ may be viewed as regularized Cauchy data obtained from the noisy Cauchy data $\left(g^{\delta}_{0}, g^{\delta}_{1}\right)$, and

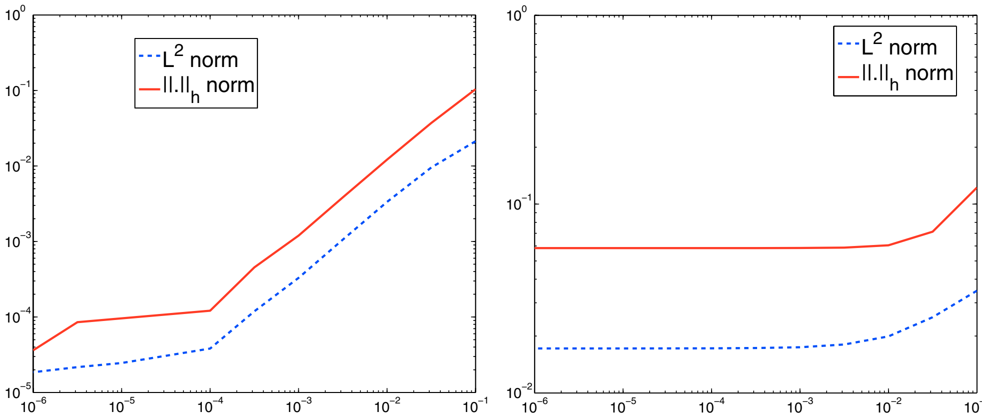

Figure 2. For $\sigma = 2\%$, left: $\|\mathrm{u}^\delta_{\alpha,h} - \mathrm{u}^\delta_h\|_h / \|\pi_h u\|_h$ and $\|\mathrm{u}^\delta_{\alpha,h} - \mathrm{u}^\delta_h\|_{L^2} / \|\pi_h u\|_{L^2}$ as a function of α; right: $\|\mathrm{u}^\delta_{\alpha,h} - \pi_h u\|_h / \|\pi_h u\|_h$ and $\|\mathrm{u}^\delta_{\alpha,h} - \pi_h u\|_{L^2} / \|\pi_h u\|_{L^2}$ as a function of α.

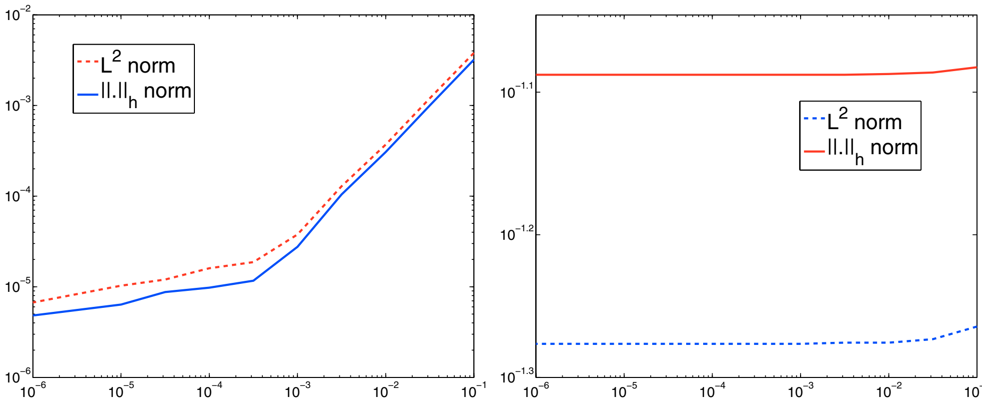

Figure 3. For $\sigma = 2\%$ and $u(x, y) = \frac{1}{50}\cos(3\pi x)\sinh(3\pi y)$, left: $\|\mathrm{u}^\delta_{\alpha,h} - \mathrm{u}^\delta_h\|_h / \|\pi_h u\|_h$ and $\|\mathrm{u}^\delta_{\alpha,h} - \mathrm{u}^\delta_h\|_{L^2} / \|\pi_h u\|_{L^2}$ as a function of α; right: $\|\mathrm{u}^\delta_{\alpha,h} - \pi_h u\|_h / \|\pi_h u\|_h$ and $\|\mathrm{u}^\delta_{\alpha,h} - \pi_h u\|_{L^2} / \|\pi_h u\|_{L^2}$ as a function of α.

this regularizing effect is actually observed in figure 4 for $\sigma = 2\%$ and in figure 5 for $\sigma = 10\%$.

In order to illustrate the second point, we verify that the value of $\varepsilon(\alpha, \delta)$ given by (7) is a good choice for ε, that is, the discrepancy between the solution u^δ_ε of (QR_ε) and the exact solution u is better with $\varepsilon = \varepsilon(\alpha, \delta)$ than with any other value of ε. In this view we begin with solving problem $(P_{\alpha,h})$ for $\alpha = 10^{-4}$, so that we obtain regularized Cauchy data $\left(\gamma_{0h}\left(\mathrm{u}^\delta_{\alpha,h}\right), \gamma_{1h}\left(\mathrm{u}^\delta_{\alpha,h}\right)\right)$, as well as $\varepsilon_h(\alpha, \delta)$ given by (7) with p^δ_α replaced by $p^\delta_{\alpha,h}$. Then we solve problem $(QR_{\varepsilon,h})$ with the previous prescribed Cauchy data $\left(\gamma_{0h}\left(\mathrm{u}^\delta_{\alpha,h}\right), \gamma_{1h}\left(\mathrm{u}^\delta_{\alpha,h}\right)\right)$ and different values of ε. We compute the corresponding errors $\left\|u^\delta_{\varepsilon,h} - \pi_h u\right\|_h$ as a function of ε. These errors are displayed in the left part of figure 7 for $\sigma = 2\%$, as well as the error obtained with $\varepsilon = \varepsilon_h(\alpha, \delta)$. It happens that such particular value of ε is almost the best possible one with the regularized data $\left(\gamma_{0h}\left(\mathrm{u}^\delta_{\alpha,h}\right), \gamma_{1h}\left(\mathrm{u}^\delta_{\alpha,h}\right)\right)$ obtained for parameter α, which is a good justification of our strategy. If we use Cauchy data $\left(\gamma_{0h}\left(\mathrm{u}^\delta_{\alpha,h}\right), \gamma_{1h}\left(\mathrm{u}^\delta_{\alpha,h}\right)\right)$ obtained with another value of

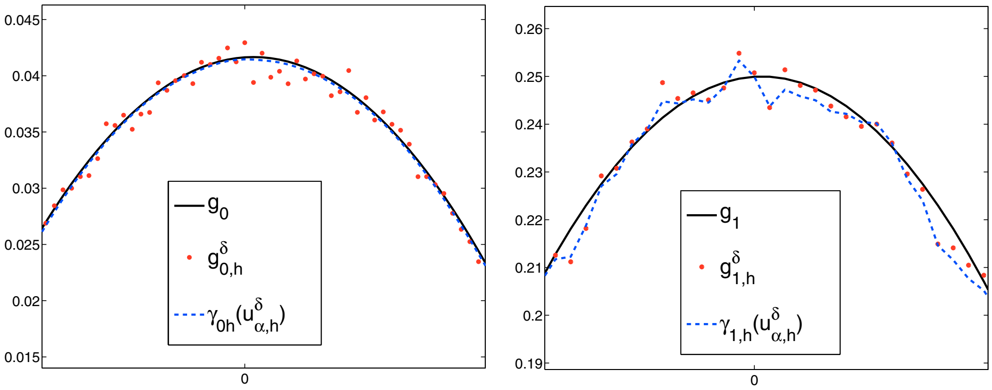

Figure 4. For $\sigma = 2\%$ and $\alpha = 10^{-4}$, left: comparison between g_0, $g^\delta_{0,h}$ and $\gamma_{0h}(\mathrm{u}^\delta_{\alpha,h})$ on the axis $y = 0.5$ around $x = 0$; right: comparison between g_1, $g^\delta_{1,h}$ and $\gamma_{1h}(\mathrm{u}^\delta_{\alpha,h})$ on the axis $y = 0.5$ around $x = 0$.

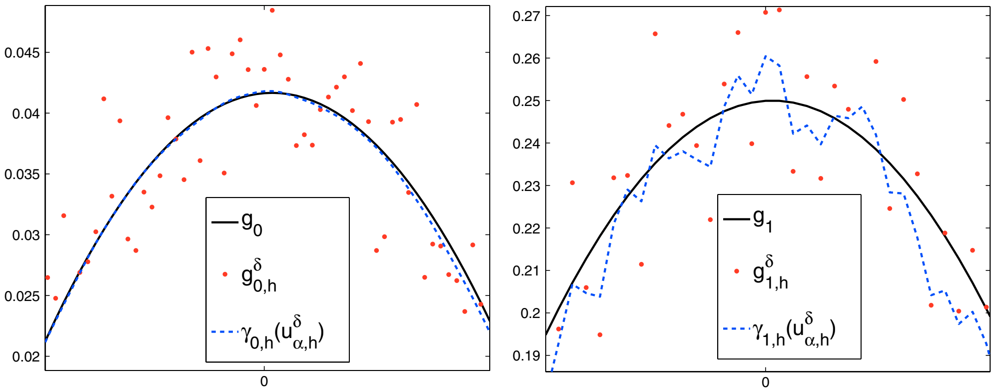

Figure 5. For $\sigma = 10\%$ and $\alpha = 10^{-4}$, left: comparison between g_0, $g^\delta_{0,h}$ and $\gamma_{0h}(\mathrm{u}^\delta_{\alpha,h})$ on the axis $y = 0.5$ around $x = 0$; right: comparison between g_1, $g^\delta_{1,h}$ and $\gamma_{1h}(\mathrm{u}^\delta_{\alpha,h})$ on the axis $y = 0.5$ around $x = 0$.

α, say $\alpha = 10^{-2}, 10^{-3}, 10^{-5}$ instead of 10^{-4}, we make the same observation in figure 6 and in the right part of figure 7, which shows a good robustness of our approach with respect to parameter α. More precisely, the curves obtained in the right part of figure 2 and in figures 6 and 7 show that in the presence of noisy data, our optimization method based on parameter α is much more robust that the original quasi-reversibility method based on parameter ε. While the error between u^δ_ε and the exact solution u strongly depends on ε, the error between u^δ_α and u is very stable on a wide range of α (in fact almost stationary as soon as α is less than 10^{-2}). We also make the same observation for $\sigma = 10\%$ in figure 8, for $\alpha = 10^{-4}$ and $\alpha = 10^{-3}$.

We also display in figure 9 the exact solution $\pi_h u$ as well as the discrepancy between the retrieved solutions $\mathrm{u}^\delta_{\alpha,h}$ and the exact solution $\pi_h u$ for various relative amplitudes of noise: $\sigma = 2\%$, 5% and 10%. The relative errors, in terms of the $\|\cdot\|_h$ norm and the L^2 norm, are given in the table below. We can observe small relative errors between the retrieved and the exact solution, even with large amplitude of noise.

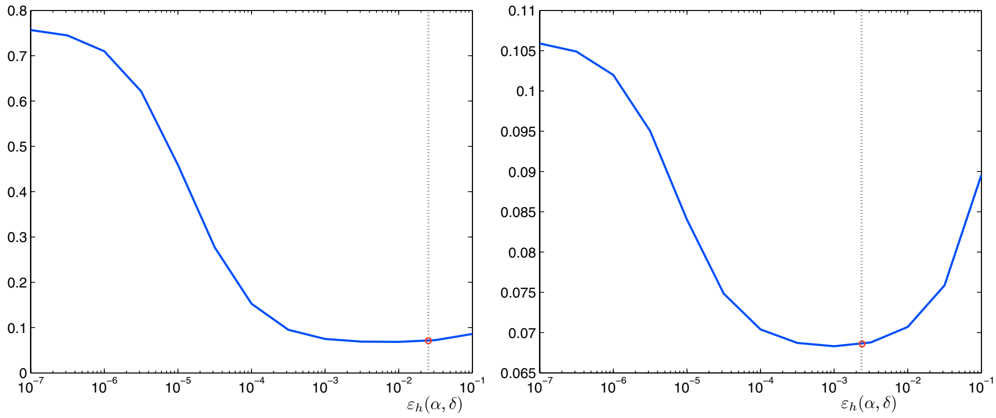

Figure 6. For $\sigma = 2\%$: $\|u^\delta_{\varepsilon,h} - \pi_h u\|_h$ as a function of ε with regularized Cauchy data $(\gamma_{0h}(\mathrm{u}^\delta_{\alpha,h}), \gamma_{1h}(\mathrm{u}^\delta_{\alpha,h}))$ for $\alpha = 10^{-2}$ (left) and $\alpha = 10^{-3}$ (right).

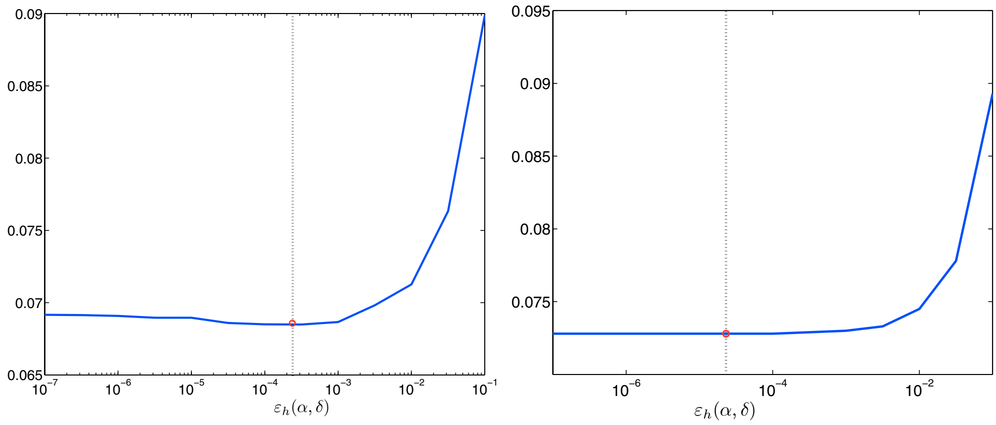

Figure 7. For $\sigma = 2\%$: $\|u^\delta_{\varepsilon,h} - \pi_h u\|_h$ as a function of ε with regularized Cauchy data $(\gamma_{0h}(\mathrm{u}^\delta_{\alpha,h}), \gamma_{1h}(\mathrm{u}^\delta_{\alpha,h}))$ for $\alpha = 10^{-4}$ (left) and $\alpha = 10^{-5}$ (right).

Relative errors	$\sigma = 2\%$	$\sigma = 5\%$	$\sigma = 10\%$
$\|\mathrm{u}^\delta_{\alpha,h} - \pi_h u\|_h / \|\pi_h u\|_h$	0.0585	0.0993	0.1427
$\|\mathrm{u}^\delta_{\alpha,h} - \pi_h u\|_{L^2} / \|\pi_h u\|_{L^2}$	0.0172	0.0336	0.0610

4.3. Application to the inverse obstacle problem

Lastly, we complete this section on numerics by emphasizing the improvement provided by our duality-based approach in the context of inverse obstacle problem. We briefly describe the inverse obstacle problem here but we refer to [8] for a detailed statement of the problem and for a description of the method that we use to solve it.

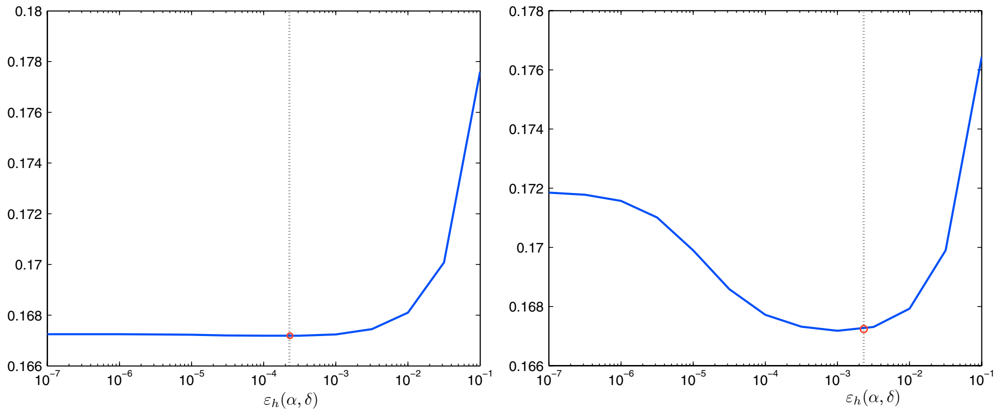

Figure 8. For $\sigma = 10\%$: $\|u^{\delta}_{\varepsilon,h} - \pi_h u\|_h$ as a function of ε with regularized Cauchy data $(\gamma_{0h}(\mathrm{u}^{\delta}_{\alpha,h}), \gamma_{1h}(\mathrm{u}^{\delta}_{\alpha,h}))$ for $\alpha = 10^{-4}$ (left) and $\alpha = 10^{-3}$ (right).

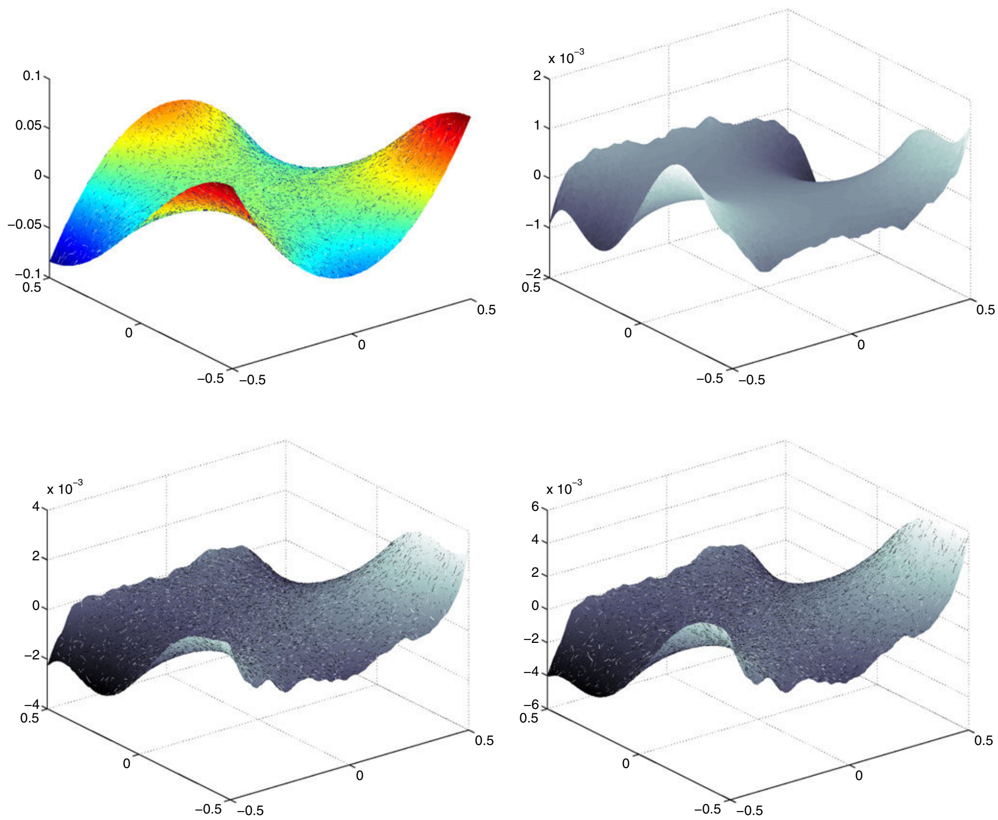

Figure 9. Top left: exact solution $\pi_h u$. Top right: for $\sigma = 2\%$ and $\alpha = 10^{-4}$, $\mathrm{u}^{\delta}_{\alpha,h} - \pi_h u$. Bottom left: for $\sigma = 5\%$ and $\alpha = 10^{-4}$, $\mathrm{u}^{\delta}_{\alpha,h} - \pi_h u$. Bottom right: for $\sigma = 10\%$ and $\alpha = 10^{-4}$, $\mathrm{u}^{\delta}_{\alpha,h} - \pi_h u$.

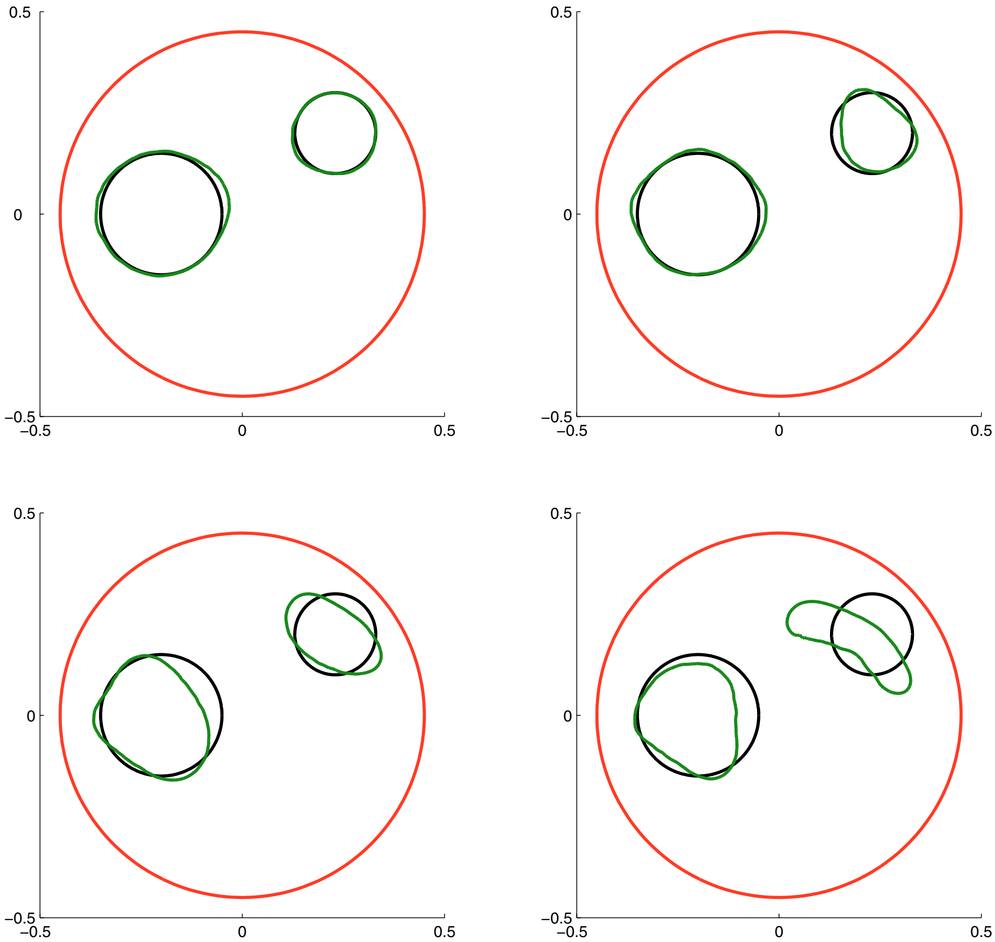

Figure 10. Top left: exact data. Top right: $\sigma = 0.5\%$. Bottom left: $\sigma = 2\%$. Bottom right: $\sigma = 10\%$.

The inverse obstacle problem consists in finding an obstacle $\mathcal{O}$ in a domain $\mathcal{D}$ from some Cauchy data (g_0, g_1) on a subpart Γ of $\partial\mathcal{D}$, such that the function u satisfies

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega \\ u = g_0 & \text{in} \quad \Gamma \\ \partial_n u = g_1 & \text{in} \quad \Gamma \\ u = 0 & \text{in} \quad \partial\mathcal{O}. \end{cases} \tag{11}$$

Starting from an initial guess $\mathcal{O}_0$ such that $\mathcal{O} \subset \mathcal{O}_0$, our method consists in finding an approximation u_m of the solution u in $\Omega_m := \mathcal{D} \setminus \overline{\mathcal{O}_m}$ with the help of the method of quasi-reversibility, and then solving in $\mathcal{O}_m$ the Poisson problem:

$$\begin{cases} \Delta v_m = f & \text{in} \quad \mathcal{O}_m \\ v_m = |u_m| & \text{in} \quad \partial\mathcal{O}_m, \end{cases} \tag{12}$$

where f has to be large enough, and finally updating the current obstacle $\mathcal{O}_m$ by defining

$$\mathcal{O}_{m+1} = \{x \in \mathcal{O}_m, v_m(x) < 0\}.$$

In [8] we provide a justification of such iterative method, in particular the fact that the sequence $(\mathcal{O}_m)$ converges in the sense of the Hausdorff distance for open domains to a set that is close to the true obstacle $\mathcal{O}$ in (11).

In the presence of noisy data $\left(g_0^\delta, g_1^\delta\right)$ instead of (g_0, g_1), we use our duality-based method to find u_0 in $\mathcal{O}_0$ (first step of the algorithm): as explained in subsection 3.4, such method provides from $\left(g_0^\delta, g_1^\delta\right)$ some regularized Cauchy data and a good choice of ε in the method of quasi-reversibility. These regularized Cauchy data and this value of ε are then our new inputs to compute the approximate solutions u_m for $m \geqslant 1$ by using the standard method of quasi-reversibility.

To illustrate the quality of results obtained with our duality-based method at the first step of our algorithm, we consider the same example as in [8], where no special procedure was used to handle noisy data. In such example, $\mathcal{D}$ is the square $]-0.5, 0.5[\times]-0.5, 0.5[$, $\mathcal{O}$ is the union of the disk of center $(-0.2, 0)$ and radius 0.15 and the disk of center $(0.23, 0.2)$ and radius 0.1, and $\Gamma = \partial\mathcal{D}$. The exact Cauchy data (g_0, g_1) we consider on $\partial\mathcal{D}$ are artificially obtained by solving a forward Laplace problem with the Dirichlet boundary condition $u = 0$ on $\partial\mathcal{O}$ and the following Neumann condition on $\partial\mathcal{D}$:

$$\begin{cases} \partial_n u = 1 & \text{on} \quad]-0.5, 0.5[\times\{-0.5\}\cup]-0.5, 0.5[\times\{0.5\} \\ \partial_n u = 0 & \text{on} \quad \{-0.5\}\times]-0.5, 0.5[\cup\{0.5\}\times]-0.5, 0.5[. \end{cases} \tag{13}$$

Some artificial noisy Cauchy data on $\partial\mathcal{D}$ with relative L^2 amplitude σ are obtained as given in (10). In our algorithm, the initial guess $\mathcal{O}_0$ is the sphere of center 0 and radius 0.45, while $f = 15$ in (12). We display in figure 10 the initial, exact and retrieved obstacles for several values of σ, namely $\sigma = 0\%$ (exact data), $\sigma = 0.5\%$, $\sigma = 2\%$ and $\sigma = 10\%$. The results are considered very good by the authors, particularly when the amplitude of noise is severe.

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